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SUR LES APPLICATIONS LINEAIRES FAIBLEMENT COMPACTES D'ESPACES DU TYPE $C(K)$

A. GROTHENDIECK

Introduction

1. Sujet. Soit K un espace compact, $C(K)$ l'espace des fonctions complexes continues sur K, muni de la norme uniforme, $\mathfrak{M}^1(K)$ son dual (espace des mesures de Radon sur K). Cet article est consacré essentiellement à l'étude des applications linéaires faiblement compactes de $C(K)$ dans des espaces localement convexes F quelconques i.e. les applications linéaires qui transforment la boule unité de $C(K)$ en une partie faiblement relativement compacte de F. Un mécanisme élémentaire de transposition montre qu'une telle étude est essentiellement équivalente à l'étude des parties faiblement relativement compactes de l'espace de Banach $\mathfrak{M}^1(K)$ (c'est à dire, si on pose $E = C(K)$, des parties de E' qui sont relativement $\sigma(E', E'')$-compactes). C'est sous cette dernière forme de la théorie que l'on saisit le mieux la raison des théorèmes qu'on obtient; à ce titre, les théorèmes 2 et 3 forment la clef de tout ce travail. Cependant, les énoncés correspondants en termes d'applications linéaires continues u de l'espace $C(K)$ dans un espace localement convexe F, sont plus frappants et plus directement adaptés aux diverses applications. Les énoncés les plus importants sont: u est faiblement compacte si et seulement si elle transforme suites faiblement convergentes en suites fortement convergentes (théorème 4); ou si elle transforme suites de Cauchy faibles en suites *faiblement* convergentes (ce qui constitue une partie du théorème 6). Les applications de ces résultats sont assez nombreuses, les plus importantes sont les théorèmes 7 et 7 bis (*toute* application linéaire continue d'un espace du type $C(K)$ dans un espace du type L^1 est faiblement compacte), le théorème 8, qui donne des propriétés remarquables d'une vaste classe d'opérations linéaires continus entre espaces localement convexes quelconques, le corollaire 2 de la proposition 10, donnant des indications tout à fait spéciales sur la "sommation vague" dans un espace de mesures de Radon. Enfin, le §4 donne des propriétés curieuses spéciales à certains espaces du type $C(K)$ particuliers: quand K est un espace compact "stonien", on obtient la propriété très spéciale indiquée dans le lemme 8 (théorème 9); quand K est homéomorphe à une suite convergente comprenant son point limite, donc si $C(K)$ est isomorphe à l'espace c_0 des suites de nombres complexes qui tendent vers zéro, nous montrons que les diverses propriétés envisagées dans les paragraphes précédents restent encore valables pour les sous-espaces et espaces quotients de $C(K) = c_0$.

Le mode d'exposition suivi est axiomatique, en ce qu'il analyse d'abord en termes abstraits les propriétés que nous avons en vue, qui servent ensuite à

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délimiter des classes remarquables d'espaces (propositions 1, 1 bis, 8, 12 et définitions 1, 2, 3, 4, 6). On voit ainsi mieux la raison des choses, et cela permet d'obtenir en passant de nombreux autres espaces jouissant de propriétés analogues à celles des espaces $C(K)$.

2. Le cadre. Ce travail aurait pu se traiter sans sortir du cadre des espaces de Banach. Comme cela n'apporterait en fait aucune simplification dans l'exposé et les démonstrations, en restreignant inutilement la portée des résultats, nous avons préféré nous placer dans le cadre des espaces localement convexes généraux. D'ailleurs, nous n'avons aucunement essayé de mettre un minimum d'hypothèses dans les énoncés. Ainsi, *toutes* les fois que nous supposons un espace localement convexe complet, il suffirait de supposer que ses parties bornées et fermées sont complètes. De même, toutes les fois que nous supposons qu'un espace est un espace du type ($\mathfrak{F}$), c'est pour pouvoir appliquer le théorème de Šmulian, alors qu'on sait que ce théorème vaut sous des conditions sensiblement plus larges (voir par exemple [**8**]). — Nous supposons connues les méthodes générales exposées dans [**3**], et ferons notamment un usage fréquent des théorèmes de Šmulian et d'Eberlein. Par ailleurs, nous employons sans références les résultats élémentaires de la théorie de la mesure, en nous plaçant exclusivement dans le cadre des mesures de Radon: une mesure sera toujours pour nous une forme linéaire sur l'espace des fonctions continues à support compact sur un espace localement compact donné. Pour les résultats moins connus sur la topologie faible dans les espaces L^1, on pourra consulter l'esposé contenu dans la première partie du travail [**4**].

3. Notations. De façon générale, nous suivons les notations des "Elements" de N. Bourbaki. Nous appelerons cependant espace précompact, tout espace uniforme A tel que l'espace uniforme séparé associé ait une complétion compacte; A peut donc ne pas être séparé. Le critère usuel de précompacité par la possibilité de recouvrements finis "arbitrairement fins", est encore valable. — Si H est un ensemble d'applications d'un espace topologique E dans un espace uniforme F, et $\mathfrak{S}$ un ensemble de parties de E, nous appellerons $\mathfrak{S}$-convergence, la structure uniforme sur H de la convergence uniforme sur les parties $A \in \mathfrak{S}$. En particulier, si A est une partie de E, la A-convergence sera la structure uniforme (non séparée en général) de la convergence uniforme sur A. H est précompact pour la $\mathfrak{S}$-convergence, si et seulement si pour tout $A \in \mathfrak{S}$, H est précompact pour la A-convergence, et H est précompact pour la A-convergence si et seulement si l'ensemble des applications de A dans F défini par les restrictions des $f \in H$ à A, est précompact pour la convergence uniforme.

Pour les espaces localement convexes, nous suivons la terminologie de [**3**]. En particulier, si E est un espace localement convexe, E' est son dual, $\langle x, x' \rangle$ désigne l'accouplement entre E et E', E'' le bidual de E' (dual de E' muni de la topologie forte). Mais il y aura lieu de considérer comme topologie "naturelle" sur E'' la topologie de la convergence uniforme sur les parties *équicontinues* de E' (et non sur les parties fortement bornées de E'). Cette topologie induit

sur E la topologie propre de E. Si E est séparé et complet, c'est un sous-espace vectoriel *fermé* de E''. — Une application linéaire d'un espace localement convexe E dans un autre F est dite *compacte* (resp. *précompacte, faiblement compacte*) si elle transforme un voisinage convenable de l'origine en une partie de F qui est relativement compacte (resp. précompacte, faiblement relativement compacte).

C désigne le corps complexe. Sauf spécification contraire, M est un espace localement compact, $C_0(M)$ l'espace des fonctions continues sur M qui "s'annulent à l'infini", muni de la norme uniforme, $\mathfrak{M}^1(M)$ son dual (espace des mesures de Radon bornées sur M) $C(M)$ l'espace de toutes les fonctions complexes continues sur M, muni de la topologie de la convergence compacte. K désignera un espace compact; alors $C_0(K) = C(K)$. Par topologie faible dans $\mathfrak{M}^1(M)$, nous entendrons la *topologie faible* définie par le dual de l'espace de Banach $\mathfrak{M}^1(M)$, en réservant le terme de *topologie vague* à la topologie faible du dual de $C_0(M)$. Quand nous parlerons de la topologie faible d'une espace L^∞ construit sur une mesure $\mathfrak{M}$, il s'agira au contraire toujours de la topologie faible du dual de L^1 (dual qu'on peut identifier à L^∞ en vertu du théorème de Lebesgue-Nikodym).

§1. Une propriété des applications linéaires faiblement compactes de certains espaces

1.1 Préliminaires. Dans [7], Dunford et Pettis prouvent, pour les espaces L^1 construits sur des cubes euclidiens, le théorème suivant: toute application linéaire faiblement compacte d'un tel espace dans un espace de Banach transforme parties faiblement compactes en parties compactes. En fait, il n'est pas difficile, soit à partir de ce résultat particulier, soit en reprenant et généralisant la méthode de Dunford-Pettis, de démontrer ce même résultat pour les espaces L^1 quelconques. Nous allons en donner ici une démonstration beaucoup plus simple et directe, ayant l'avantage de mettre en évidence les propriétés de L^1 qui interviennent réellement dans le théorème envisagé. Cela nous permettra en même temps d'établir le même théorème pour d'autres catégories importantes d'espaces, et notamment les espaces $C(K)$ (espace des fonctions continues sur un espace compact K, muni de la topologie de la convergence uniforme), et de reconnaître que ce théoréme est lié à une propriété qui semble nouvelle (prop. 2).

Nous commençons par établir trois lemmes élémentaires sur les espaces localement convexes généraux, généralisant des propriétés bien connues des applications linéaires faiblement compactes ou compactes entre espaces de Banach, et que nous utiliserons constamment par la suite.

Lemme 1. *Soit u une application linéaire d'un espace localement convexe E dans un autre F, supposé séparé. Les quatre conditions suivantes sont toutes équivalentes:*

(1) *u transforme les parties bornées de E en parties faiblement relativement compactes de F.*

(2) *La bitransposée u'' de u applique E'' dans F.*

(3) *u'' transforme les parties équicontinues de E'' (considéré comme dual de E' fort) en parties faiblement relativement compactes de F.*

(4) *La transposée u' de u est une application continue de F' muni de la topologie de Mackey $\tau(F', F)$ dans E' fort (ou aussi: de F' faible dans E' muni de la topologie $\sigma(E', E'')$).*

Cela implique:

(5) *u' transforme les parties faiblement compactes de F' en parties $\sigma(E', E'')$-compactes de E'.*

Réciproquement, si F est complet, pour que les conditions équivalentes 1 *à* 4 *soient vérifiées, il suffit déjà qu'on ait*

(5 bis) *u' transforme les parties équicontinues et faiblement fermées de F' en parties $\sigma(E', E'')$-compactes de E'.*

(*Donc si F est complet*, toutes *les conditions précédentes sont équivalentes.*)

Démonstration. Rappelons que E'' est la réunion des adhérences faibles dans E'' des parties bornées de E, adhérences qui sont d'ailleurs faiblement compactes et constituent un système fondamental de parties équicontinues de E''; et que u'' s'obtient en prolongeant u en une application de E'' dans F'' par continuité faible. Si donc u transforme les parties bornées A de E en parties faiblement relativement compactes de F, elle transforme leurs adhérences faibles dans E'' en parties faiblement relativement compactes *dans* F, (1) implique donc (2) et (3). Réciproquement, (2) et *a fortiori* (3) implique (1), u'' étant faiblement continue et les parties équicontinues de E'' faiblement relativement compactes. Donc les conditions (1) à (3) sont équivalentes. De plus, le premier énoncé dans (4) signifie que tout ensemble équicontinu de formes linéaires sur E' fort donne, en composant avec u', un ensemble équicontinu de formes linéaires sur F' muni de $\tau(F', F)$, c'est à dire précisément (3). De même, le deuxième énoncé dans (4) est manifestement équivalent à (2), ce qui prouve que toutes les conditions de (1) à (4) sont équivalentes. Le deuxième énoncé dans (4) implique évidemment (5). Reste à voir que (5 bis) implique (2) quand F est complet; mais (5 bis) signifie aussi que u'' est continue quand E'' est muni de la topologie $\tau(E'', E')$, et F'' de la topologie de la convergence uniforme sur les parties équicontinues de F', — topologie qui induit sur F sa topologie propre. Or E est dense dans E'' pour $\tau(E'', E')$, puisque toute forme linéaire continue sur E'' qui s'annule sur E est nulle, et F étant complet, est un sous-espace *fermé* de F''. Comme u'' applique E dans F, il suit par continuité que u'' applique E'' dans F.

Lemme 2. *Soit une application linéaire continue d'un espace localement convexe E dans un autre F, $\mathfrak{S}$ un ensemble de parties bornées de E. Les conditions suivantes sont équivalentes:*

(1) *Pour tout $A \in \mathfrak{S}$, $u(A)$ est précompact.*

(2) *Pour toute partie équicontinue B' de F', $u'(B')$ est une partie de E' précompacte pour la $\mathfrak{S}$-convergence.*

Démonstration. (1) signifie aussi que la transposée u' est continue, quand

on munit F' de la topologie T_c de la convergence uniforme sur les parties précompactes de F, et E' de la topologie $T(\mathfrak{S})$ de la convergence uniforme sur les éléments de $\mathfrak{S}$. Comme toute partie équicontinue de F' est précompacte pour la topologie T_c (conséquence bien connue du théorème d'Ascoli) il suit bien que (1) implique (2). On en déduit aussitôt que (2) implique (1), en désignant par $\mathfrak{T}$ l'ensemble des parties équicontinues de F', et en échangeant les rôles de u, u' et $\mathfrak{S}$, $\mathfrak{T}$.

Remarque. Si la topologie $T(\mathfrak{S})$ est séparée sur E' (c'est à dire si la réunion des $A \in \mathfrak{S}$ est totale dans E) alors on peut dans l'énoncé (2) remplacer le mot "précompact" par "relativement compact", comme il résulte aussitôt de la démonstration (les parties équicontinues et faiblement fermées de F' étant T_c-compactes).

LEMME 3. *Soient E, E' deux espaces vectoriels en dualité faible, A une partie faiblement bornée de E, A' une partie faiblement bornée de E'. Les quatre conditions suivantes sont équivalentes:*

(1) *A' est précompact pour la A-convergence.*

(1') *A est précompact pour la A'-convergence.*

(2) *Sur A', la structure uniforme faible est plus fine que la structure uniforme de la A-convergence.*

(2') *Sur A, la structure uniforme faible est plus fine que la structure uniforme de la A'-convergence.*

De plus, si A (resp. A') est convexe symétrique, on peut remplacer dans l'énoncé (2') (resp. dans 2) les structures uniformes par les topologies correspondantes.

De toutes façons, les conditions équivalentes précédentes impliquent que l'ensemble des $\langle x, x'\rangle$ ($x \in A$, $x' \in A'$) est borné, et que la fonction $\langle x, x'\rangle$ sur

$$A \text{ faible} \times A' \text{ faible}$$

est uniformément continue.

Enfin, si E est un espace $(\mathfrak{F})$ et si A est faiblement compact, les conditions équivalentes précédentes sont aussi équivalentes à:

(3) *Toute suite faiblement convergente extraite de A converge uniformément sur A'.*

Démonstration. On a (1) → (2'), car si A'_A désigne A' muni de la structure uniforme de la A-convergence, l'ensemble $\tilde{A}$ des fonctions sur A'_A définies par les $x \in A$ est par définition même uniformément équicontinu, on sait donc que sur $\tilde{A}$, la structure uniforme de la convergence simple (dans A'_A) est identique à la structure uniforme de la convergence uniforme sur l'espace *précompact* A'_A, d'où résulte bien (2').

On a (2') → (1'), car A est précompact pour la structure uniforme faible, et *a fortiori* pour toute structure uniforme moins fine.

Par symétrie, on aura donc aussi (1') → (2) et (2) → (1), ce qui établit l'équivalence des quatre conditions envisagées. L'uniforme continuité de $\langle x, x'\rangle$ sur A faible $\times A'$ faible en résulte immédiatement, car $\langle x, x'\rangle$ est de toutes façons uniformément continu sur le produit $A_{A'} \times A'_A$, or ici A faible

est plus fin que $A_{A'}$, A' faible plus fin que A'_A. Il en suit aussi que l'ensemble des $\langle x, x' \rangle$ ($x \in A, x' \in A'$) est partie précompacte, donc bornée, du corps des scalaires, comme image de l'espace précompact A faible $\times A'$ faible par une application uniformément continue.

Le fait que l'on peut dans l'énoncé (2) remplacer les structures uniformes par les topologies lorsque A est convexe symétrique résulte du facile résultat suivant:

Soit A une partie convexe symétrique d'un espace vectoriel E, T et T' deux topologies localement convexes sur E (ici, la topologie de la A'-convergence, resp. la topologie faible de E). *Alors, pour que T et T' induisent sur A la même structure uniforme il suffit déjà qu'elles induisent le même système de voisinages de zéro.*

En effet, un système fondamental d'entourages de la structure uniforme induite par T, par exemple, est obtenu en prenant, pour tout voisinage V de zéro pout T, l'ensemble des $(x, y) \in A \times A$ tels que $x - y \in V$. Or, A étant convexe symétrique, on aura $x - y \in 2A$, de sorte que la relation ci-dessus s'écrit aussi $x - y \in 2A \cap V$, soit aussi $\frac{1}{2}(x - y) \in A \cap \frac{1}{2}V$. Donc un système fondamental d'entourages est obtenu en prenant pour tout voisinage de zéro U induit par T sur A, l'ensemble des $(x, y) \in A \times A$ tels que $\frac{1}{2}(x - y) \in U$. Le résultat annoncé en résulte aussitôt.

Le lecteur remarquera que dans les démonstrations précédentes les structures vectorielles n'interviennent pas en fait (sauf pour la dernière).

Il est évident que (2′) implique (3); réciproquement, si E est du type $(\mathfrak{F})$ et si A est faiblement compact, (3) implique (1′). Il suffit de montrer que de toute suite extraite de A on peut extraire une suite qui converge uniformément sur A', mais d'après le théorème de Šmulian, on peut en effet extraire une suite qui converge *faiblement*.

Remarque. Soit $\mathfrak{S}$ un ensemble de parties faiblement bornées A de E, A' une partie faiblement bornée de E'. On sait que, pour que A' soit précompact pour la $\mathfrak{S}$-convergence, il faut et il suffit que pour tout $A \in \mathfrak{S}$, A' soit précompact pour la A-convergence. Le lemme 3 donne alors diverses autres conditions équivalentes, que nous ne repèterons pas. Si on suppose donné aussi un ensemble $\mathfrak{S}'$ de parties faiblement bornées de E', alors le lemme 3 donne l'équivalence des conditions suivantes: (1) Tout $A' \in \mathfrak{S}'$ est précompact pour la $\mathfrak{S}$-convergence; (2) sur tout $A' \in \mathfrak{S}'$, la structure uniforme faible est plus fine que la structure uniforme de la $\mathfrak{S}$-convergence; et les deux énoncés symétriques. Enfin, dans le cas où E est un espace $(\mathfrak{F})$, on peut énoncer aussi le

COROLLAIRE. *Soit E un espace $(\mathfrak{F})$, $\mathfrak{S}$ un ensemble de parties faiblement compactes de E, A' une partie bornée de E'. Alors A' est précompact pour la $\mathfrak{S}$-convergence si et seulement si toute suite faiblement convergente extraite d'un $A \in \mathfrak{S}$ converge uniformément sur A'. En particulier, une partie A' de E' est relativement compacte pour la topologie $\tau(E', E)$ si et seulement si toute suite faiblement convergente dans E converge uniformément sur A'* (c'est à dire, dans la terminologie de G. Köthe, si A' est "*begrenzt*" dans la dualité avec E).

La dernière partie du corollaire résulte du théorème de Krein (l'enveloppe convexe cerclée fermée d'un faiblement compact est faiblement compacte).

1.2 La propriété de Dunford-Pettis.

PROPOSITION 1. *Soit E un espace localement convexe, $\mathfrak{S}$ un ensemble de parties bornées de E. Les hypothèses suivantes sont équivalentes:*

(1) *Pour tout espace localement convexe séparé F, toute application linéaire continue de E dans F qui transforme parties bornées en parties faiblement relativement compactes, transforme les $A \in \mathfrak{S}$ en parties précompactes;*

(2) *Même énoncé, mais F étant supposé un espace de Banach;*

(3) *Les $A \in \mathfrak{S}$ sont précompacts pour la topologie de la convergence uniforme sur les parties de E' équicontinues, convexes, cerclées et $\sigma(E', E'')$-compactes; ou une des formes équivalentes, d'après le lemme* 3, *à la condition* (3), *en particulier:*

(4) *Toute partie équicontinue convexe cerclée et $\sigma(E', E'')$-compacte de E' est précompacte pour la $\mathfrak{S}$-convergence.*

Démonstration. On a évidemment (1) → (2).

(2) → (3). Soit E_T l'espace E muni de la topologie T de la convergence uniforme sur les parties équicontinues, convexes, cerclées et $\sigma(E', E'')$-compactes de E'. Comme tout espace localement convexe, E_T est isomorphe à un sous-espace vectoriel d'un produit topologique d'espaces de Banach F_i, de telle sorte que pour établir qu'une partie $A \in \mathfrak{S}$ est précompacte dans E_T, il suffit de montrer que toute application linéaire continue u de E_T dans un espace de Banach F transforme A en une partie précompacte de F (en fait, ce raisonnement suppose E_T séparé, mais on se libère trivialement de cette restriction). Mais u sera *a fortiori* application continue de E dans F, et sa transposée u', application linéaire de F' dans E', applique de plus les parties équicontinues de F' en parties de E' qui sont équicontinues en tant que parties du dual de E_T, donc $\sigma(E', E'')$-compactes. Donc (lemme 1) u transforme parties bornées de E en parties faiblement relativement compactes de F, donc par hypothèse transforme $A \in \mathfrak{S}$ en une partie précompacte de F.

(3) → (1). On peut évidemment supposer l'espace F de l'énoncé de (1) *complet*. Soit u une application linéaire continue de E dans F transformant parties bornées en parties faiblement rel. compactes, soit $A \in \mathfrak{S}$; il faut montrer que $u(A)$ est précompact. Mais A étant par hypothèse précompact pour la topologie T définie plus haut (topologie qui donne encore pour dual E' d'après le théorème de Mackey), il suffit de prouver que u est continue de E_T dans F. Il revient au même de dire que u' transforme parties équicontinues convexes cerclées faiblement compactes de F' en parties équicontinues du dual de E_T, c'est à dire en parties équicontinues et $\sigma(E', E'')$-compactes du dual de E. Mais cela est vrai, car u étant continue de E dans F, u' transforme parties équicontinues de F' en parties équicontinues de E', qui de plus sont ici relativement $\sigma(E', E'')$-compactes en vertu du lemme 1.

Définition 1. *On dit que l'espace localement convexe E jouit de la propriété de Dunford-Pettis (en abrégé, propriété* D.-P.), *s'il satisfait à l'une des hypothèses équivalentes de la proposition* 1, *où $\mathfrak{S}$ désigne l'ensemble des parties de E convexes cerclées et faiblement compactes.*

Cela signifie donc aussi que les applications linéaires continues de E dans un espace localement convexe séparé quelconque qui transforment les ensembles bornés en parties faiblement relativement compactes, transforment les parties faiblement compactes convexes cerclées de E en parties précompactes, donc compactes puisqu'elles sont faiblement compactes (et par suite complètes); ou encore que les parties équicontinues, convexes, cerclées, $\sigma(E', E'')$-compactes de E' sont aussi $\tau(E', E)$-compactes. (Voir remarques plus bas.)

Corollaire de la proposition 1. *Si E' fort jouit de la propriété* D.-P., *il en est de même de E, pourvu que les parties fortement bornées de E' soient équicontinues (en particulier, si E est un espace* ($\mathfrak{F}$))

En effet, grace à la forme (4) de l'hypothèse D.-P. sur E' fort, cette hypothèse s'énonce; les parties convexes cerclées A du dual E'' de E' fort qui sont équicontinues (i.e. contenues dans l'adhérence faible dans E'' d'une partie bornées de E) et $\sigma(E'', E''')$-compactes, sont compactes pour la topologie $\tau(E'', E')$. Ici, E'' désigne le dual fort de E' fort. Mais E'' induisant sur E la topologie de E par hypothèse, l'énoncé précédent vaut en particulier si A est une partie convexe cerclée et faiblement compacte de E, ce qui établit le corollaire en vertue de la forme (3) de la propriété D.-P.

Remarques. (a) Dans les énoncés (1) à (3), on peut remplacer le mot "précompact" par "compact" si on suppose les $A \in \mathfrak{S}$ faiblement compacts. En effet, on sait qu'une partie précompacte et faiblement compacte d'un espace vectoriel localement convexe F est compacte, car une partie faiblement compacte est complète (fait bien connu, et qui se vérifie immédiatement en considérant E comme un espace de formes linéaires sur E'). Notons aussi que dans le cas où la réunion des $A \in \mathfrak{S}$ est partie totale de E, donc que sur E' la topologie de la $\mathfrak{S}$-convergence est séparée, (4) signifie aussi que toute partie convexe, cerclée, équicontinue et $\sigma(E', E'')$-compacte de E' est *compacte* pour la topologie de la $\mathfrak{S}$-convergence.

(b) Rappelons que d'après un résultat de Krein pour espaces de Banach, généralisé dans [**9**] aux espaces localement convexes quelconques, l'enveloppe convexe cerclée fermée d'une partie faiblement compacte est elle-même faiblement compacte, si (et seulement si) elle est complète (pour la topologie donnée sur E), en particulier chaque fois que E est complet. Cela permet, si E est complet, de supprimer le mot "convexe cerclé" dans l'explicitation que nous avons donné après la définition 1. Cela permet aussi, dans tous les cas, de supprimer dans les énoncés (3) et (4) de la proposition 1, l'hypothèse que les parties envisagées de E' sont convexes cerclées; car je dis que si A' est une partie équicontinue, $\sigma(E', E'')$-compacte de E', son enveloppe convexe cerclée fortement fermée est encore $\sigma(E', E'')$-compacte, et évidemment équicontinue.

En effet, il suffit de voir que cette enveloppe est fortement complète, or il est bien connu, et trivial à vérifier, que toute partie faiblement relativement compacte et fortement fermée de E' est fortement complète. Bien entendu, si E est un espace $(\mathfrak{F})$, on peut aussi supprimer dans les énoncés (3) et (4) l'hypothèse que les parties envisagées de E' sont équicontinues, puisqu'elles le sont automatiquement en tant que parties bornées.

Une propriété, analogue à la propriété D.-P. et plus forte dans les cas usuels, est mise en évidence dans la

PROPOSITION 1 BIS. *Soit E un espace localement convexe. Les hypothèses suivantes sont équivalentes:*

(1) *Pour tout espace localement convexe F, toute application linéaire continue u de E dans F qui transforme les parties bornées en parties faiblement relativement compactes, transforme les suites de Cauchy faibles en suites de Cauchy fortes dans F.*

(2) *Même énoncé, mais F étant supposé un espace de Banach.*

(3) *Toute suite de Cauchy faible dans E est suite de Cauchy pour la topologie de la convergence uniforme sur les parties équicontinues, convexes cerclées et $\sigma(E', E'')$-compactes de E'.*

(4) *Les parties de E' équicontinues, convexes, cerclées et $\sigma(E', E'')$-compactes sont compactes pour la topologie de la convergence uniforme sur les suites de Cauchy faibles de E.*

La démonstration est identique à celle de la proposition 1. Il suffit de remarquer qu'une suite de Cauchy faible est aussi suite de Cauchy pour la topologie envisagée dans (3), si et seulement si elle est précompacte pour cette topologie (lemme 3). — Notons aussi que l'énoncé (1) est équivalent au suivant: l'application envisagée u transforme parties bornées et faiblement métrisables A de E en parties précompactes de F (car c'est manifestement suffisant, réciproquement, comme A est de toutes façons faiblement précompact, de toute suite extraite de A on pourra extraire une suite de Cauchy faible, dont l'image sera suite de Cauchy forte dans F si on suppose (1) vérifié; mais d'après un résultat bien connu de A. Weil, cela implique que $u(A)$ est partie précompacte de F). — La proposition précédente conduit à poser la

DÉFINITION 2. *On dit que l'espace localement convexe E jouit de la propriété de Dunford-Pettis stricte (en abrégé, propriété* D.-P. *stricte) si les conditions équivalentes de la proposition* 1 *bis sont vérifiées.*

Si E est un espace $(\mathfrak{F})$, cette propriété est plus forte que la propriété D.-P. (lemme 3, corollaire). Les deux propriétés sont équivalentes si dans l'espace $(\mathfrak{F})$ envisagé, les suites de Cauchy faibles sont faiblement convergentes, par exemple pour les espaces L^1.

Nous verrons plus bas des applications intéressantes des propriétés D.-P. et D.-P. stricte, et notamment du fait que les espaces $C(K)$ et L^1 les possèdent (théorème 1). Signalons seulement pour l'instant que, si E jouit de la propriété

D.-P., et si u et v sont des endomorphismes de E transformant parties bornées en parties faiblement relativement compactes, alors l'application linéaire composée uv transforme parties bornées en parties relativement compactes. Si par exemple E est un espace de Banach, cela donne: le produit de deux endomorphismes faiblement compacts de E est un endomorphisme compact (résultat déjà signalé par Dunford et Pettis pour des espaces L^1). Il suit que la théorie de Riesz des endomorphismes compacts peut alors s'étendre aux endomorphismes faiblement compacts. — Notons en passant qu'un espace de Banach réflexif ne peut avoir la propriété D.-P. que s'il est de dimension finie, car sa boule unité devrait être précompacte.

Une autre propriété intéressante des espaces qui jouissent de la propriété D.-P., et qui ne semble pas avoir été mise en évidence jusqu'ici même dans le cas particulier des espaces $C(K)$ et L^1 est donnée par la

PROPOSITION 2. *Soient E et F deux espaces localement convexes complets, G un espace localement convexe, u une application bilinéaire continue de $E \times F$ dans G. Soit (x_i) une suite dans E faiblement convergente vers un $x_0 \in E$, (y_i) une suite dans F faiblement convergente vers un $y_0 \in F$. Si E jouit de la propriété* D.-P., *alors $u(x_i, y_i)$ tend faiblement vers $u(x_0, y_0)$.*

Démonstration. La conclusion signifie que pour tout $z' \in G'$, $\langle u(x_i, y_i), z'\rangle$ tend vers $\langle u(x_0, y_0), z'\rangle$, de sorte qu'on est ramené au cas où G est le corps de base R ou C (en considérant la forme bilinéaire $\langle u(x, y), z'\rangle$). Mais alors, il est bien connu et trivial qu'il existe une application linéaire continue v de F dans E' fort telle que $u(x, y) = \langle x, vy\rangle$. La suite des $x'_i = vy_i$ converge vers $x'_0 = vy_0$ au sens de $\sigma(E', E'')$, il faut montrer que $\langle x_i, x'_i\rangle$ tend vers $\langle x_0, x'_0\rangle$. En vertu de la forme (3) de la propriété D.-P. (proposition 1) il suffit de montrer que l'enveloppe convexe cerclée fermée de l'ensemble des x_i (respectivement des x'_i dans E' faible) est faiblement compacte dans E (resp. $\sigma(E', E'')$-compacte). Pour la première, cela résulte de l'extension du théorème de Krein que nous avons déjà signalé après la définition 1; pour la seconde, il suffit d'appliquer le théorème de Krein *dans* F à la suite (y_i), et de conclure que l'image par v de l'enveloppe convexe cerclée fermée des y_i est $\sigma(E', E'')$-compacte.

Bien entendu, on aurait un énoncé légèrement renforcé si on suppose que E jouit de la propriété D.-P. stricte. Les réciproques sont d'ailleurs vraies quand E est un espace de Banach, comme il est facile de s'en convaincre en prenant pour u l'accouplement canonique entre E et E'. Si en effet E ne jouissait pas de la propriété, D.-P., il existerait une suite (x_i) dans E faiblement convergente vers zéro, et une partie $\sigma(E', E'')$-compacte A' de E', telles que (x_i) ne converge pas uniformément sur A'. On pourrait donc trouver $\epsilon > 0$, et une suite (x'_i) dans A', tels que $|\langle x_i, x'_i\rangle| > \epsilon$, et par extraction d'une suite partielle, on pourrait supposer que la suite (x'_i) est convergente pour $\sigma(E', E'')$ (grâce au théorème de Šmulian), ce qui établit notre assertion.

On voit aussi par là que la proposition 2 donne une propriété assez particulière aux hypothèses envisagées, puisqu'elle est déjà en défaut par exemple pour la

forme bilinéaire canonique entre un espace de Banach de dimension infinie réflexif, et son dual.

1.3 Cas des espaces $C(K)$ **et** L^1. Nous arrivons au résultat essentiel de ce paragraphe (la deuxième partie du théorème étant essentiellement connue):

THÉORÈME 1. *L'espace E jouit de la propriété* D.-P. *stricte* (voir définitions 1 et 2) *dans chacun des deux cas suivants:*

(a) *E est un espace* $C(K)$ (espace des fonctions continues sur un espace compact K, muni de la topologie de la convergence uniforme).

(b) *E est un espace* L^1 *construit sur une mesure arbitraire.*

Démonstration. Comme dans un espace L^1 les suites de Cauchy faibles sont faiblement convergentes, il suffit de démontrer la propriété D.-P. pour L^1 pour prouver (b). Mais le dual de L^1 est L^∞ (théorème de Lebèsgue-Nikodym), il est donc isomorphe à un espace $C(K)$ d'après un théorème bien connu de Gelfand, de sorte qu'en vertu du corollaire de la proposition 1, nous pouvons nous borner à établir le théorème pour l'espace $C(K)$. Soit donc (f_i) une suite de Cauchy faible dans $C(K)$, c'est donc une suite uniformément bornée, et qui converge en chaque point $t \in K$ vers une limite $g(t)$ (la réciproque étant d'ailleurs vraie en vertu du théorème de Lebèsgue). g est une fonction borélienne et bornée sur K, donc intégrable pour toute mesure de Radon sur K. Nous voulons montrer que $\langle f_i, \mu\rangle$ tend vers une limite uniformément quand μ parcourt un ensemble A' de mesures de Radon compact pour la topologie $\sigma(E', E'')$. D'ailleurs, en vertu du théorème de Lebèsgue, cette limite ne peut être que l'intégrale $\int g\, d\mu$; si la convergence n'était pas uniforme, on pourrait de A' extraire une suite (μ_i) telle que sur cette suite, la convergence ne soit déjà pas uniforme. Mais il est bien connu que pour une *suite* de mesures de Radon sur K, il existe une mesure de Radon positive μ telle que toutes les μ_i de la suite appartiennent à l'espace des mesures définies par les éléments de $L^1(\mu)$. Comme $L^1(\mu)$ s'identifie, avec sa norme, à un sous-espace vectoriel fermé de l'espace $\mathfrak{M}^1(K)$ de toutes les mesures sur K, la suite (μ_i) s'identifie ici à une suite faiblement relativement compacte de $L^1(\mu)$, soit (h_i). On est donc ramené à montrer que $\int f_i(t)h(t)d\mu(t)$ tend vers $\int g(t)h(t)d\mu(t)$ uniformément quand h parcourt une partie faiblement compacte de $L^1(\mu)$. Mais cela résulte du théorème d'Egoroff et du résultat plus fort suivant, qui a son intérêt propre:

PROPOSITION 3. *Soit μ une mesure de Radon positive sur l'espace localement compact M. Si la suite (f_i) dans $L^\infty(\mu)$ est uniformément bornée, et converge en mesure sur tout compact $K \subset M$ vers une $g \in L^\infty$, alors elle converge vers g au sens de la topologie $\tau(L^\infty, L^1)$, i.e. $\int f_i h d\mu$ tend vers $\int gh\, d\mu$ uniformément quand h parcourt une partie faiblement compacte de* L^1 (pour une réciproque, voir corollaire 1 de la proposition 4).[1]

Démonstration. Soit A une partie faiblement compacte de L^1, et $\epsilon > 0$.

[1]Cette proposition et sa démonstration m'ont été suggérés par M. Dieudonné.

On sait (voir [4], théorème 4) qu'il existe un compact $K \subset M$ et un $\eta > 0$, tels que $h \in A$ implique

(a)
$$\int_{\complement K} |h| \, d\mu < \epsilon,$$

(b) pour toute partie mesurable B de M telle que $\mu(B) < \eta$,

$$\int_B |h| \, d\mu < \epsilon.$$

Supposons $\|f_i\|^\infty < R$ pour tout i, on a alors, pour $h \in A$,

$$\left| \int f_i h d\mu - \int g h d\mu \right| = \left| \int (f_i - g) h \, d\mu \right| < \left| \int_{\complement K} (f_i - g) h \, d\mu \right| + \left| \int_K (f_i - g) h \, d\mu \right|.$$

Le premier terme est $< 2R\epsilon$. D'après l'hypothèse, la suite (f_i) converge uniformément vers g dans un compact $K_0 \subset K$ tel que $\mu(K \cap \complement K_0) < \eta$. Le deuxième terme est donc majoré par

$$2R\epsilon + \int_{K_0} |f_i - g| \, |h| \, d\mu$$

qui est $< 2R\epsilon + R'a_i$, où

$$R' = \sup_{h \in A} \|h\|_1, \quad a_i = \sup_{t \in K_0} |f_i(t) - g(t)|.$$

Comme a_i tend vers zéro pour i infini, on aura donc

$$\left| \int f_i h \, d\mu - \int g h \, d\mu \right| < 5R\epsilon \quad \text{pour } i \text{ assez grand,}$$

ce qui établit la proposition 3.

Remarquons qu'en fait, la proposition 3, ainsi que ses corollaires, pourrait s'énoncer pour un filtre sur une partie bornée de L^∞, convergeant en mesure sur tout compact.

Corollaire 1 de la proposition 3. *Soit u une application linéaire faiblement continue (voir Introduction) de L^∞ dans un espace localement convexe séparé F. Alors u est faiblement compacte, et transforme toute suite de L^∞ qui est bornée et qui converge en mesure sur tout compact vers $g \in L^\infty$, en une suite fortement convergente (i.e. convergente pour la topologie donnée sur F) vers $u(g)$. Même conclusion si u est une application faiblement continue de L^∞ dans le dual G' d'un espace localement convexe G, qui transforme la boule de L^∞ en une partie $\sigma(E', E'')$-compacte de G'.*

En effet, dans le premier cas par exemple, comme la boule de L^∞ est faiblement compacte, il suite d'abord que son image par u est faiblement compacte, d'où (lemme 1) que u' transforme parties équicontinues de F' en parties faiblement

relativement compactes de L^1, d'où résulte facilement la conclusion en vertu de la proposition 3. La deuxième partie du corollaire se démontre de façon analogue.

Un cas particulier du corollaire 1 est le corollaire suivant, qui généralise un théorème bien connu d'*Orlicz:*

COROLLAIRE 2. *Soit $\Phi(t)$ une application faiblement sommable de M dans un espace localement convexe F, (i.e. telle que pour $x' \in F'$, $\langle \Phi(t), x' \rangle$ soit fonction sommable) et telle que $\int \Phi f \, d\mu$ soit élément de F quelle que soit $f \in L^\infty$. Alors l'application $f \to \int \Phi f d\mu$ est faiblement compacte, et transforme toute suite bornée de L^∞ qui converge en mesure sur tout compact vers une $g \in L^\infty$, en suite fortement convergente vers $\int \Phi g \, d\mu$*

En effet, d'après la définition même de l'intégrale faible, l'application $f \to \int \Phi f \, d\mu$ est faiblement continue. — Une autre application analogue et plus profonde sera donnée au §2, proposition 10. Comparez aussi avec §2, théorème 5.

Donnons en passant une autre forme de la proposition 3:

PROPOSITION 4. *Soit μ une mesure de Radon positive sur l'espace localement compact M. Pour qu'une partie A de $L^\infty(\mu)$ soit $\tau(L^\infty, L^1)$-compacte, il faut et il suffit qu'elle soit bornée, et que pour tout compact $K \subset M$, l'ensemble A_K des restrictions à K des $f \in A$ soit partie compacte de L^1* (la restriction de f à K signifie ici la fonction nulle dans $\complement K$ qui coïncide avec f dans K).

Démonstration. (a) Nécessité. Soit L_K^∞ l'ensemble des fonctions éléments de L^∞ qui sont nulles dans $\complement K$. Comme l'application: "restriction à K", est manifestement une application linéaire de L^∞ dans lui-même continue pour $\tau(L^\infty, L^1)$, A_K sera aussi partie τ-compacte de L_K^∞. Comme la boule B_K de L_K^∞ est partie faiblement compacte de L^1 (voir [**4**], théorème 4) A_K est donc aussi compact pour la topologie de la convergence uniforme sur B_K, considéré comme partie de L^1. Mais cette topologie sur L_K^∞ n'est autre que la topologie induite par L^1.

(b) Suffisance. Supposons que A satisfasse aux conditions de la propostion 4, montrons que A est $\tau(L^\infty, L^1)$-compact. Il est d'abord très facile de vérifier que A est partie faiblement fermée donc faiblement compacte de L^∞, il suffit donc de montrer maintenant que si un filtre $\mathfrak{F}$ sur A converge faiblement vers g, il converge même uniformément sur toute partie B de L^1 qui est faiblement compacte. Mais A étant borné dans L^∞(soit $\|f\|_\infty < R$ si $f \in A$), on voit aussitôt qu'il suffit de vérifier que pour tout compact $K \subset M$, la restriction f_K de f à K converge vers g_K uniformémént sur B, suivant le filtre $\mathfrak{F}$; car on pourra trouver, pour $\epsilon > 0$ donné, un compact K tel que $h \in B$ implique

$$\int_{\complement K} |h| \, d\mu < \epsilon,$$

d'où

$$\left| \int_{\complement K} (f - g) h \, d\mu \right| < 2R\epsilon$$

pour $f \in A$. Nous sommes donc ramenés au cas où M est un compact K, et où A est une partie bornée de L^∞ compacte pour la topolgie T induite par L^1 Pour prouver qu'alors A est τ-compact, il suffit de montrer que l'application identique de A muni de T dans L^∞ muni de τ est continue. Comme A muni de T est métrisable, cela revient à dire que si la suite (f_i) extraite de A converge vers une limite g au sens de T, elle converge au sens de τ. Mais ce n'est autre que la proposition 3 (car on sait que (f_i) convergera aussi vers g en mesure).

Corollaire. *La condition suffisante donnée dans la proposition* 3 *pour qu'une suite dans L^∞ converge pour $\tau(L^\infty, L^1)$ est aussi nécessaire.* (Car une suite convergente pour τ est relativement compacte pour τ, d'autre part pour une suite uniformément bornée dans un espace L^∞ construit sur un compact, convergence au sens de L^1 et convergence en mesure sont identiques.)

Nous laissons au lecteur le soin d'énoncer l'analogue des corollaires 1 et 2 de la proposition 3, relativement à la proposition 4.

1.4 Application à d'autres espaces vectoriels. A partir du théorème 1, on obtient d'autres espaces vectoriels qui jouissent de la propriété D.-P. stricte. Tout d'abord, on a le

Corollaire du théorème 1. *Si M est un espace localement compact, l'espace $\mathfrak{M}^1(M)$ des mesures de Radon bornées sur M jouit de la propriété* D.-P. *stricte.*

En effet, d'après un théorème de S. Kakutani **[11]**, $\mathfrak{M}^1(M)$ est isomorphe à un espace L^1. — En particulier, le dual $\mathfrak{M}^1(K)$ d'un espace $C(K)$ est isomorphe à un espace L^1; à cette occasion, signalons qu'inversement, comme nous l'avons rappelé plus haut, le dual d'un espace L^1 est isomorphe à un espace $C(K)$. Cette dualité remarquable entre les espaces du type L^1 et espaces du type $C(K)$ est souvent utile, et nous servira encore dans la suite de cet article.

D'autres applications du théorème 1 sont obtenues à l'aide de la

Proposition 5. *Un facteur direct d'un espace qui jouit de la propriété* D.-P. (*resp. de la propriété* D.-P. *stricte*) *jouit de la même propriété. Le produit vectoriel topologique d'une famille d'espaces qui jouissent de la propriété* D.-P. (*resp. de la propriété* D.-P. *stricte*) *jouit de la même propriété.*

La démonstration est triviale.—On fera attention qu'on n'a rien d'analogue pour les sous-espaces ou les espaces quotients, même dans le cas des espaces de Banach; on sait en effet que *tout* espace de Banach est isomorphe (avec sa norme) à un sous-espace d'un espace $C(K)$, et à un espace quotient d'un espace L^1. Cette réflexion nous donne un grand nombre d'exemples où un sous-espace vectoriel fermé d'un espace $C(K)$ où L^1 n'a pas de supplémentaire, puisqu'on a le

Corollaire. *Un espace de Banach qui ne satisfait pas à la propriété D.P. stricte ne peut être isomorphe à un facteur direct d'un espace L^1 ou $C(K)$. En*

particulier, il en est ainsi pour les espaces de Banach réflexifs de dimension infinie.

Généralisons d'abord le théorème 1 à une classe analogue d'espaces.

Proposition 6. (a) *Soit M un espace normal, $\mathfrak{S}$ un ensemble de parties de M, $C(M, \mathfrak{S})$ l'ensemble des fonctions continues sur M et bornées sur les $A \in \mathfrak{S}$, muni de la topologie de la $\mathfrak{S}$-convergence. Cet espace jouit de la propriété* D.-P. *et de la propriété* D.-P. *stricte.*

(b) *Soit M un espace localement compact muni d'une mesure μ, $\mathfrak{S}$ un ensemble de parties mesurables et relativement compactes de M, soit $\mathfrak{L}(\mu,\mathfrak{S})$ l'espace des fonctions localement sommables sur M, muni de la topologie de la convergence en moyenne sur les $A \in \mathfrak{S}$. Cet espace jouit de la propriété* D.-P. *et de la propriété* D.-P. *stricte.*

Démontrons par exemple (a), la démonstration est toute analogue pour (b). Supposons les $A \in \mathfrak{S}$ fermés et l'ensemble $\mathfrak{S}$ filtrant, ce qui est loisible. Il faut montrer que si un filtre $\mathfrak{F}$ dans $E = C(M, \mathfrak{S})$ est soit le filtre élémentaire associé à une suite de Cauchy faible, soit un filtre faiblement convergent sur une partie faiblement compacte de E, alors ce filtre converge uniformément sur toute partie A' du dual E' qui est équicontinue et $\sigma(E', E'')$-compacte. A' étant partie équicontinue, il existe un voisinage V de zéro dans E, défini par exemple par: $|f(t)| < 1$ pour $t \in A$ (A étant une partie fixée de M, élément de $\mathfrak{S}$) sur lequel les $\mu \in A'$ soient majorés en module par une constante k. Soit $f \to uf$ l'opération de restriction à A; M étant normal, c'est une application linéaire continue de $C(M, \mathfrak{S})$ *sur* l'espace $F = C^\infty(A)$ des fonctions continues *bornées* sur A (muni de la norme de la convergence uniforme), et qui applique V *sur* la boule de $C^\infty(A)$. Sa transposée u' est donc une application linéaire continue *biunivoque* de F' dans E', appliquant la boule de F' *sur* le polaire V° de V. Donc toute $\mu \in A'$ provient d'une $\tilde{\mu} \in F'$ et d'une seule, Montrons que l'ensemble $\tilde{A}$ de ces $\tilde{\mu}$ est une partie de F' qui est $\sigma(F', F'')$-compacte, il suffit pour ce de montrer que u' est isomorphisme fort de F' dans E', ou, ce qui revient au même, qu'il existe une partie bornée B de E dont l'image par u soit dense dans la boule de F. Mais en vertu du théorème d'Urysohn, on peut en effet prendre pour B l'ensemble des $f \in E$ qui sont en module inférieures à 1 sur tout M. — Pour montrer maintenant que

$$\lim_{\mathfrak{F}} \langle f, \mu \rangle$$

existe, uniformément quand μ parcourt A', on écrit $\langle f, \mu \rangle = \langle uf, \tilde{\mu} \rangle$, et on note que, puisque u est continue, l'image de $\mathfrak{F}$ par u est un filtre dans F qui est soit filtre élémentaire associé à une suite de Cauchy faible dans F, soit filtre faiblement convergent sur une partie faiblement compacte de F. Il suffit maintenant d'appliquer le théorème 1 a), car on sait que $C^\infty(A)$ s'identifie à l'espace $C(K)$ sur le compactifiée de Stone-Čech K de l'espace A [**14**].

Proposition 7. *Soit O un ouvert de $\mathbf{R}^n$, et soit $\mathfrak{E}^{(m)}(O)$ l'espace des fonctions dans O, m fois continûment différentiables, muni de la topologie de la convergence*

compacte de f et de ses dérivées partielles jusqu'à l'ordre m. Cet espace jouit de la propriété D.-P. *stricte. Il en est de même de l'espace analogue* $\mathfrak{E}^{(m)}(K)$ *construit sur un cube compact* K *de* $\mathbf{R}^n$.

Moyennant la proposition 5 et 6, cela résulte du

LEMME 4. *Considérons* $\mathfrak{E}^{(m)}(O)$ *comme sous-espace du produit vectoriel topologique*

$$\prod_{|p|\leq m} C_p$$

d'espaces $(\mathfrak{F})$ *tous identiques à l'espace* $C(O)$ *des fonctions continues sur* O, *muni de la topologie de la convergence compacte, - grace à l'isomorphisme* $f \to (D^p f)_{|p|\leq m}$ *de* $\mathfrak{E}^{(m)}(O)$ *dans ce produit* ($p = (p_1, \ldots, p_n)$ désignant un indice de dérivation multiple d'ordre $|p|$,

$$D^p = \partial^p/\partial x_1^{p_1} \ldots \partial x_n^{p_n}).$$

Alors il est facteur direct dans ce produit. Enoncé analogue pour l'espace $\mathfrak{E}^{(m)}(K)$ *construit sur un cube compact* K *de* $\mathbf{R}^n$, *quand cet espace est considéré comme sous-espace du produit topologique*

$$\prod_{|p|\leq m} C_p$$

d'espaces tous identiques à $C(K)$.

Démonstration. Donnons la par exemple pour l'espace $\mathfrak{E}^{(m)}(K)$.[2] Soit 0 un point de l'intérieur du cube K, soit E_0 le sous-espace de $E = \mathfrak{E}^{(m)}(K)$ formé des f telles que $D^p f(0) = 0$ pour $|p| \leq m - 1$. C'est un sous-espace vectoriel de E de codimension finie, il suffit donc de montrer que E_0 est facteur direct dans

$$\prod_{|p|\leq m} C_p$$

[2] *Note ajoutée pendant la correction des épreuves.* La démonstration du lemme 4 dans le cas d'un ouvert O n'est aussi simple que si O est par exemple convexe. Sinon, la démonstration est un peu plus compliquée, mais il n'est pas difficile de se ramener "au local" par les méthodes standart de partitions de l'unité.

Remarquons aussi que dans le cas où K est un cube compact, on a même un résultat plus fort que le fait que $\mathfrak{E}^{(m)}(K)$ soit isomorphe à un facteur direct d'un espace $C(K')$. En effet, *alors* $\mathfrak{E}^{(m)}(K)$ *est isomorphe à* $C(K)$. La démonstration est facile si K est le segment $K_0 = (0, 1)$: alors, si H est le sous-espace de codimension 1 de $\mathfrak{E}^{(m)}(K_0)$ formé des fonctions nulles en 0 ainsi que leurs dérivées d'ordre $\leq m - 1$, l'application $f \to f^{(m)}$ de H dans $C(K_0)$ est évidemment un isomorphisme du premier espace sur le second; d'autre part il est bien connu que la somme directe de $C(K_0)$ avec un espace de dimension 1, donc aussi avec un espace de dimension finie m, est isomorphe à $C(K_0)$. D'autre part, si le théorème annoncé est vrai pour un compact K, il en résulte facilement que pour tout espace de Banach E, les espaces de Banach $\mathfrak{E}^{(m)}(K, E)$ et $C(K, E)$ sont isomorphes (car isomorphes resp. à l'espace des applications linéaires compactes et faiblement continues de E' dans $\mathfrak{E}^{(m)}(K)$ et $C(K)$). Faisant en particulier $E = \mathfrak{E}^{(m)}(K_0) \approx C(K_0)$, on trouve que $\mathfrak{E}^{(m)}(K \times K_0)$ est isomorphe à $C(K \times K_0)$. Le théorème annoncé apparait maintenant par récurrence sur la dimension de K. — Le théorème analogue est vrai, avec la même démonstration, pour R^n au lieu de K. J'ignore si on peut même remplacer R^n par n'importe quelle variété indéfiniment différentiable.

c'est à dire qu'il existe une projection continue u de $\prod C_p$ sur E_0. On posera $u((f_p)_p) = (g_p)_p$, les g_p étant définies en fonction du système $(f_p)_p$ de la façon suivante:

$$g_p = f_p \qquad \text{si } |p| = m,$$

$$g_p(x) = \int_{0x} g_{p+\alpha_1}(t_1)\, dt_1 + \ldots + g_{p+\alpha_n}(t_n)\, dt_n \qquad \text{si } |p| < m,$$

où α_i désigne l'indice de dérivation correspondant à l'opérateur $\partial/\partial x_i$ (toutes ses composantes sont nulles, sauf la *i*éme, égale à 1), et $0x$ est le segment orienté joignant 0 à x. (Rappelons qu'on ajoute les indices de dérivation en ajoutant leurs composantes.) Il est évident de proche en proche que les g_p sont des applications linéaires continues de $\prod C_p$ dans C_p. De plus on aura $g_p(0) = 0$ pour $|p| < m$, et

$$\frac{\partial}{\partial x_i} g_p = g_{p+\alpha_i}.$$

Il en résulte que u est bien une projection continue sur E_0.

Remarques. Bien entendu, le même raisonnement vaut pour l'espace $\mathfrak{C}^{(m)}(K)$ construit sur un compact K quelconque, pourvu que sa frontière ne soit pas trop compliquée; où pour l'espace des fonctions m fois continûment différentiables sur une variété m fois différentiable, et les espaces de champs de tenseurs analogues, par les méthodes standard de passage au local. Il n'est peut-être pas inutile de remarquer, en vue de l'application de la proposition 7, que l'on a la caractérisation suivante des suites de Cauchy faibles (et partant, des suites faiblement convergentes) dans $\mathfrak{C}^{(m)}(O)$ et $\mathfrak{C}^{(m)}(K)$, critère résultant immédiatement de l'immersion dans le produit topologique d'espaces $C(O)$ (resp. $C(K)$): (f_i) est suite de Cauchy faible dans $\mathfrak{C}^{(m)}(O)$ (resp. $\mathfrak{C}^{(m)}(K)$) si et seulement si elle est bornée, et si pour tout $t \in O$ (resp. tout $t \in K$) et tout indice de dérivation multiple p d'ordre $\leqslant m$, la suite des $D^p f_i(t)$ est convergente. La suite (f_i) converge faiblement vers g si et seulement si on a de plus $\lim D^p f_i(t) = D^p g\,(t)$.

Un exemple. Comme nous l'avons remarqué, un sous-espace vectoriel fermé d'un espace de Banach jouissant de la propriété D.-P., ou un espace quotient d'un tel espace, peut ne pas jouir de la propriété D.-P. On pourrait supposer toute fois qu'on ait l'énoncé suivant: Tout espace quotient d'un espace $C(K)$ jouit de la propriété D.-P. D'après le corollaire de la proposition 1, un énoncé équivalent serait: tout sous-espace vectoriel fermé d'un espace L^1 jouit de la propriété D.-P. Cela impliquerait que tout sous-espace vectoriel réflexif d'un espace L^1 est de dimension finie. Or, comme me l'a fait remarquer M. P. Malliavin, il n'en est rien, car il existe des sous-espaces vectoriels fermés H de dimension infinie de l'espace L^1 construit sur le segment $(0, 1)$, tels que toute $f \in H$ soit dans L^2; comme alors l'application identique de H dans L^2 est continue pour la topologie induite sur H par L^1 (grâce au théorème "du graphe fermé") et que d'autre part la topologie induite par L^2 sur H est plus fine que celle induite par L^1, il suit que ces deux topologies sont en fait identiques,

donc que H est sous-espace vectoriel réflexif de dimension infinie de L^1.—Quant à la construction de H, il suffit, de la suite des $\phi_n(t) = e^{2i\pi n t}$, d'extraire une sous-suite (ϕ_{n_k}), avec $n_{k+1}/n_k \geqslant \lambda > 1$, et de prendre pour H le sous-espace fermé de L^1 engendré par les ϕ_{n_k}. Alors toute $f \in H$ a une série de Fourier lacunaire, d'où suit en vertu d'un résultat classique (voir Zygmund, Trigonometrical Series, page 122) que $f \in L^2$.

§2. Critères de faible compacité dans les espaces L^1 et les espaces de mesures de Radon bornées, et une propriété réciproque de la propriété D.-P.

2.1 Critères de faible compacité dans $\mathfrak{M}^1(M)$. Soit M un espace localement compact, $C_0(M)$ l'espace des fonctions complexes continues sur M qui "s'annulent à l'infini", $\mathfrak{M}^1(M)$ son dual, c'est à dire l'espace des mesures de Radon bornées sur M. Si μ est une mesure positive quelconque sur M, et si à toute $\phi \in L^1(\mu)$ on fait correspondre la mesure $\phi\, d\mu$, on sait qu'on obtient un isomorphisme normé de $L^1(\mu)$ dans $\mathfrak{M}^1(M)$. Il suit que tout critère de compacité ou de faible compacité dans les espaces $\mathfrak{M}^1(M)$ donne un critère correspondant dans les espaces L^1 (dans la suite, par *topologie faible* dans $M^1(M)$ nous entendons la topologie faible définie par la dualité avec le dual de cet espace de Banach; nous appelerons *topologie vague* la topologie faible du dual de $C_0(M)$).[3]

Ce paragraphe est destiné essentiellment à l'exposé de critères de faible compacité dans $\mathfrak{M}^1(M)$. La proposition 3 du paragraphe précédent donne une condition nécéssaire remarquable pour qu'une partie A d'un espace L^1 soit faiblement relativement compacte. Nous allons transformer cette condition en une condition qui s'énonce directement pour une partie A d'un espace $\mathfrak{M}^1(M)$, puis nous montrerons que diverses formes en apparence plus faibles de cette condition suffisent déjà pour impliquer la faible compacité relative. La condition (4) du théorème qui suit (la plus faible de toutes en apparence) m'a été suggérée par la lecture de **[5]** (voir loc. cité, proposition 5).

Théorème 2. *Soit M un espace localement compact, A une partie bornée de l'espace $\mathfrak{M}^1(M)$ des mesures de Radon bornées sur M. Pour que A soit faiblement relativement compacte, il faut et il suffit qu'elle satisfasse à l'une des conditions suivantes, toutes équivalentes:*

(1) *Pour toute suite uniformément bornée (f_i) de fonctions complexes sur M, mesurables pour toute $\mu \in A$, et qui convergent en chaque point vers une fonction $g(t)$, on a*

$$\lim_i \int f_i\, d\mu = \int g\, d\mu\,,$$

[3] L'inverse ne semble pas vrai, puisque le critère de Dunford-Pettis (voir **[4]**, théorème 4) s'énonce en faisant intervenir de façon essentielle la mesure μ qui sert de base. Néanmoins, en se servant de l'isomorphisme de $\mathfrak{M}^1(M)$ à un espace L^1 (**[11]**), et de la condition nécessaire du critère de Dunford-Pettis, il n'est pas difficile de voir que si A est partie faiblement compacte de $\mathfrak{M}^1(M)$, il existe une $\mu \in \mathfrak{M}^1(M)$ telle que $A \subset L^1(\mu)$, ce qui permettrait d'interprèter certains énoncés pour espaces L^1, en énoncés analogues pour espaces $\mathfrak{M}^1(M)$.

uniformément quand μ parcourt A. — Dans cet énoncé, on peut aussi se borner à supposer que pour tout compact $K \subset M$ et toute $\mu \in A$, f_i tend vers g en mesure sur K, relativement à la mesure μ.

(2) *Pour toute suite (f_i) faiblement convergente vers zéro dans $C_0(M)$*, (i.e. uniformément bornée, et tendant vers zéro en tout point), *on a*

$$\lim_i \langle f_i, \mu\rangle = 0$$

uniformément quand μ parcourt A.

(3) *Pour toute suite (O_i) d'ouverts disjoints deux à deux, on a*

$$\lim_i \mu(O_i) = 0$$

uniformément quand μ parcourt A.

(4) *On a l'ensemble des deux conditions:* (a) *Pour tout compact $K \subset M$ et tout $\epsilon > 0$, existe un voisinage ouvert U de K tel que $|\mu|(U \cap \complement K) < \epsilon$ pour toute $\mu \in A$.* (b) *Pour tout $\epsilon > 0$ existe un compact $K \subset M$ tel que $|\mu|(\complement K) < \epsilon$ pour toute $\mu \in A$.*

Démonstration. Démontrons d'abord la nécessité de la condition (1), en en prenant tout de suite la forme la plus forte. Si (1) n'était pas vérifié, il existerait un $\epsilon > 0$, une suite (g_n) extraite de la suite donnée et une suite (μ_n) extraite de A, telles que

$$\left| \int (g_n - g)\, d\mu_n \right| > \epsilon$$

pour tout n. Soit

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_n ,$$

il est immédiat de vérifier que g_n tend vers g en mesure pour μ sur tout compact. Mais la suite (μ_n) peut être regardée comme une suite faiblement relativement compacte dans $L^1(\mu)$, de sorte que nous aurions contradiction avec la proposition 3 du §1.

Evidemment (1) implique (2). Prouvons que (2) implique (3). Si en effet (3) n'était pas vérifié, alors, en extrayant au besoin de (O_i) une suite partielle, il existerait $\epsilon > 0$ et une suite (μ_i) extraite de A tels que $|\mu_i(O_i)| > \epsilon$ pour tout i. Mais pour tout i existe alors une fonction continue positive f_i à support compact contenu dans O_i, et partout < 1, telle que $|\int f_i \, d\mu_i| > \epsilon$, ce qui contredit la condition (2).

(3) implique (4), car si par exemple la condition (4), a, n'était pas vérifiée, on construirait par récurrence une suite décroissante (V_n) de voisinages ouverts de K, une suite (μ_n) extraite de A et une suite (O_n) d'ouverts relativement compacts, tels que

$$\bar{O}_n \subset V_{n-1} \cap \complement V_n, \quad |\mu_n(O_n)| > \tfrac{1}{2}\epsilon$$

(ce qui contredit manifestement la condition (3)). Supposons la construction

faite jusqu'au rang n, on peut alors continuer d'un rang de la façon suivante. Il existe par hypothèse $\mu_{n+1} \in A$ telle que $|\mu_{n+1}(V_n \cap \complement K)| > \epsilon$, donc un compact $K_{n+1} \subset V_n \cap \complement K$ tel que $|\mu_{n+1}(K_{n+1})| > \frac{1}{2}\epsilon$, donc un ouvert relativement compact O_{n+1} tel que $K_{n+1} \subset O_{n+1} \subset \bar{O}_{n+1} \subset V_n \cap \complement K$ et $|\mu_{n+1}(O_{n+1})| > \frac{1}{2}\epsilon$. Il suffit maintenant de prendre un voisinage ouvert V_{n+1} de K contenu dans V_n et ne rencontrant pas le compact $\bar{O}_{n+1}$.—On établit de façon identique la condition (4), b.

Reste à prouver que (4) implique que A est faiblement relativement compact. En vertu du théorème d'Eberlein, il suffit de montrer que toute suite (μ_i) extraite de A admet un point d'adhérence faible; mais on sait que la suite des μ_i est contenu dans un même espace $L^1(\mu)$, de sorte qu'on est ramené à prouver que les conditions correspondantes aux conditions (a), (b), ci-dessus *dans un espace* L^1, impliquent que A est faiblement relativement compact.[4] Mais ici, en vertu du critère classique de Dunford-Pettis, il suffit de montrer que pour tout $\epsilon > 0$, existe un $\eta > 0$ tel que pour toute partie ouverte U de M de mesure $< \eta$, on ait

$$\int_U |f|\, d\mu < \epsilon, \qquad \text{pour } f \in A.$$

On peut d'ailleurs, grâce à la condition b, supposer que U est astreint à rester dans un compact fixe, de sorte que nous sommes ramenés au cas où M est compact. — Procédons alors par l'absurde; si A n'était pas faiblement relativement compact, il existerait une suite (U_n) de parties ouvertes de M, et une suite (f_n) extraite de A, telles que $\mu(U_n) < 2^{-n}$,

$$\int_{U_n} |f_n|\, d\mu > \epsilon.$$

Posons

$$V_n = \bigcup_{i \geq n} U_i$$

les V_n forment une suite décroissante d'ouverts dont les mesures tendent vers zéro, et on a

$$\int_{V_n} |f_n|\, d\mu > \epsilon.$$

Notons qu'en passant aux complémentaires, la condition (a) implique que pour tout ouvert V et tout $\alpha > 0$, existe un compact $K \subset V$ tel que

$$\int_{V \cap K} |f|\, d\mu < \alpha, \qquad \text{pour } f \in A.$$

Soit donc K_n un compact contenu dans V_n tel que

[4]Comme $L^1(\mu)$ est un sous-espace vectoriel fermé de $\mathfrak{M}^1(M)$, sa topologie faible est la topologie induite par la topologie faible de $\mathfrak{M}^1(M)$, et $L^1(\mu)$ est faiblement fermé. Par suite, une partie de $L^1(\mu)$ y est faiblement relativement compacte si et seulement si elle est faiblement relativement compacte dans $\mathfrak{M}^1(M)$.

$$\int_{V_n \cap K_n} |f|\, d\mu < 2^{-n-1}\epsilon, \qquad \text{pour } f \in A.$$

Soit

$$K'_n = \bigcap_{1 \leqslant i \leqslant n} K_i$$

on a $K'_n \subset K_n \subset V_n$, et

$$\int_{K'_n} |f_n|\, d\mu \geqslant \int_{V_n} |f_n|\, d\mu - \int_{V_n \cap \complement K'_n} |f_n|\, d\mu.$$

Or

$$V_n \cap \complement K'_n = \bigcup_{1 \leqslant i \leqslant n} (V_n \cap \complement K_i) \subset \bigcup_{1 \leqslant i \leqslant n} (V_i \cap \complement K_i),$$

d'où

$$\int_{V_n \cap \complement K'_n} |f_n|\, d\mu < \sum_{1 \leqslant i \leqslant n} 2^{-i-1}\epsilon < \tfrac{1}{2}\epsilon,$$

donc

$$\int_{K'_n} |f_n|\, d\mu \geqslant \epsilon - \tfrac{1}{2}\epsilon = \tfrac{1}{2}\epsilon.$$

Mais les K'_n forment une suite décroissants de compacts, évidemment non vides, dont les mesures tendent vers zéro. Leur intersection K n'est pas vide, et on a $\mu(K) = 0$. Par hypothèse, il existe un voisinage ouvert U de K tel que

$$\int_{U \cap \complement K} |f|\, d\mu < \tfrac{1}{2}\epsilon, \qquad \text{pour } f \in A,$$

soit, puisque K est négligeable,

$$\int_U |f|\, d\mu < \tfrac{1}{2}\epsilon.$$

Mais K étant l'intersection d'une suite décroissante de compacts K'_n, un des compacts K'_n est contenu dans U. On aurait alors, puisque

$$\int_{K'_n} |f_n|\, d\mu \geqslant \tfrac{1}{2}\epsilon\,,$$

l'inégalité

$$\int_U |f_n|\, d\mu \geqslant \tfrac{1}{2}\epsilon\,,$$

ce qui amène contradiction et par suite démontre le théorème.

Corollaire du théorème 2. *Soit $E = C_0(M)$, $E' = \mathfrak{M}^1(M)$, A une partie de E'. Pour que A soit relativement $\sigma(E', E'')$-compacte, il faut et il suffit que l'on ait la condition*

(2) bis. *A est relativement compact pour $\tau(E', E)$.*

En effet, on sait (corollaire du lemme 3 du §1) que cette condition est équivalente à la condition (2) du théorème 2.

Remarque. Le résultat essentiel de [**5**] (loc. cité, prop. 8)—et qui nous servira encore par la suite—affirme que, si M est compact métrisable, *une suite* (μ_i) *dans* $\mathfrak{M}^1(M)$ *converge faiblement si et seulement si pour tout ouvert O dans M, la suite des* $\mu_i(O)$ *est convergente.* A titre didactique, montrons comment ce résultat peut se déduire assez simplement du théorème 2, $\mathfrak{M}$ étant un espace localement compact quelconque (pas forcément métrisable). Il suffit manifestement de montrer que sous les conditions ci-dessus, la suite (μ_i) est faiblement relativement compacte. On montre assez simplement (loc. cité, prop. 9) que la suite reste bornée, il suffit donc de montrer que pour toute suite (O_j) d'ouverts deux à deux disjoints, on a

$$\lim_j \mu_i(O_j) = 0$$

uniformément en i. Soit l^1 l'espace des suites sommables de nombres complexes avec sa topolgie naturelle, et pour toute $\mu \in \mathfrak{M}^1(M)$ soit $\tilde{\mu} \in l^1$ définie par $\tilde{\mu}(j) = \mu(O_j)$. On vérifie immédiatement que la suite $(\tilde{\mu}_i)$ est suite de Cauchy faible dans l^1 (car elle est bornée, et pour toute partie N_1 de l'ensemble N des entiers,

$$\sum_{j \in N_1} \tilde{\mu}_i(i) = \mu_i(\bigcup_{j \in N_1} O_j)$$

tend vers une limite), donc faiblement convergente, ce qui implique (d'après le critère de Dunford-Pettis par exemple) que

$$\lim_j \tilde{\mu}_i(j) = 0$$

uniformément en i, cqfd.

Le résultat de Dieudonné que nous venons de rappeler permet de prouver un intéressant complément au théorème 2. Pour l'énoncer, notons d'abord que si $\mathfrak{L}^\infty(M)$ désigne l'espace des fonctions bornées sur M qui sont mesurables pour toute mesure de Radon sur M, alors $\mathfrak{L}^\infty(M)$ muni de la norme uniforme s'identifie à un sous-espace du dual de $\mathfrak{M}^1(M)$, toute $f \in \mathfrak{L}^\infty(M)$ définissant la forme linéaire continue $\mu \rightarrow \int f \, d\mu$. Cela étant, on a le

THÉORÈME 3. *Soit M un espace localement compact, β_0 l'espace des fonctions sur M engendré par les fonctions caractéristiques des parties fermées de M. Pour qu'une partie bornée A de $\mathfrak{M}^1(M)$ soit faiblement relativement compacte, il suffit (et il faut évidemment) qu'elle satisfasse à la condition suivante*

(5) *A est relativement compact pour la topologie $\sigma(\mathfrak{M}^1, \beta_0)$*

Démonstration. On se ramène d'abord au cas où M est compact de la façon suivante. Si M n'était pas compact, soit $\hat{M}$ l'espace compact obtenu en adjoignant à M son "point à l'infini" ω; on sait qu'alors $\mathfrak{M}^1(M)$ s'identifie à l'hyperplan fortement fermé de $\mathfrak{M}^1(\hat{M})$ formé des mesures de Radon sur $\hat{M}$ qui n'ont pas de masse au point ω. On est donc ramené à montrer que A est partie faiblement

relativement compacte de $\mathfrak{M}^1(\hat{M})$, or il est immédiat de vérifier que A satisfait encore en tant que partie de $\mathfrak{M}^1(\hat{M})$ à la condition du théorème 3.

Supposons donc M compact, et envisageons d'abord le cas où M est de plus métrique. — Comme A est borné, sur l'adhérence vague de A, la topologie $\sigma(\mathfrak{M}^1, \beta_0)$ coïncide avec la topologie $\sigma(\mathfrak{M}^1, \bar{\beta}_0)$, où $\bar{\beta}_0$ est l'adhérence forte de β_0 dans le dual de $\mathfrak{M}^1(M)$, ou, ce qui revient au même, son adhérence dans $\mathfrak{L}^\infty(M)$. Cette adhérence contient $C(M)$, comme il est bien connu, et facile à voir. D'après le théorème d'Eberlein, il suffit de montrer que de toute suite extraite de A on peut extraire une suite faiblement convergente. Mais M étant métrique, donc $C(M)$ séparable, donc la boule unité de $\mathfrak{M}^1(M)$ vaguement métrisable, on peut de la suite donnée extraire une suite qui converge vaguement. D'après les remarques précédentes et parce que A est relativement compact pour $\sigma(\mathfrak{M}^1, \bar{\beta}_0)$, cette suite converge donc aussi au sens de $\sigma(\mathfrak{M}^1, \bar{\beta}_0)$ donc aussi au sens de la topologie faible de $\mathfrak{M}^1(M)$, en vertu du théorème de Dieudonné rappelé plus haut. — Le cas où M est un compact quelconque se ramène au cas métrique grace au

LEMME 5. *Soit M un espace compact, A un ensemble borné de mesures de Radon sur M. Pour que A soit partie faiblement relativement compacte de $\mathfrak{M}^1(M)$, il faut et il suffit que pour tout espace quotient séparé métrisable $\tilde{M}$ de M, l'image canonique de A dans $\mathfrak{M}^1(\tilde{M})$ soit une partie faiblement relativement compacte de cet espace.*

Démonstration. On peut évidemment se borner à établir la suffisance de la condition, et supposer A convexe, cerclé, vaguement fermé. Signalons alors le lemme facile et bien connu:

LEMME 6. *Si E est un espace localement convexe, A une partie équicontinue convexe, cerclée et faiblement fermée de son dual, il existe une application linéaire continue u de E dans espace de Banach F telle que la transposée u' applique biunivoquement la boule unité de F' sur A.*

Pour le voir, il suffit de considérer la semi-norme sur E définie par l'ensemble polaire A° de A, et d'appeler F l'espace de Banach déduit de cette semi-norme de la façon usuelle (en faisant un quotient pour obtenir une vraie norme, et en complétant). L'application u naturelle de E dans F satisfait à la condition voulue.

Dans le cas actuel, pour démontrer que A est partie du dual E' de $E = C(M)$, compacte pour $\sigma(E', E'')$, il suffit de démontrer que l'application u définie dans le lemme précédent est faiblement compacte (lemme 1) donc, en vertu du théorème d'Eberlein, que pour toute suite (f_i) extraite de la boule de $C(M)$, la suite des $u(f_i)$ a une valeur d'adhérence faible dans F. Mais soit $\tilde{M}$ l'espace quotient de M par la relation d'équivalence "$f_i(s) = f_i(t)$ pour tout i". $\tilde{M}$ est manifestement séparé donc compact, et de plus métrique, car sa topologie est la topologie la moins fine de celles qui rendent continues les fonctions déduites des f_i par passage au quotient (topologie en effet moins fine, et encore séparée).

$C(\tilde{M})$ s'identifie à un sous-espace vectoriel de $C(M)$, contenant les f_i. Tout revient à montrer que la restriction v de u à $C(\tilde{M})$ est faiblement compacte, c'est à dire (lemme 1) que l'image de la boule unité de F' par v' est partie du dual $H' = \mathfrak{M}^1(\tilde{M})$ de $H = C(\tilde{M})$ qui est relativement compacte pour $\sigma(H', H'')$. Or, on vérifie immédiatement que cette image n'est autre que l'image canonique de A dans $\mathfrak{M}^1(\tilde{M})$, ce qui établit le lemme 5.

Remarque. Le lemme 5 permet d'affaiblir encore certains criteres suffisants de relative faible compacité dans $\mathfrak{M}^1(M)$. Par exemple, si M est compact (pour simplifier), on peut dans l'énoncé du théorème 3 remplacer l'espace β_0 par l'espace engendré par les fonctions caracteristiques de parties fermées ayant une *suite* fondamentale de voisinage (ce ne sont autres que les images réciproques de parties fermées dans des quotients séparés métriques de M). La modification analogue vaudrait pour les énoncés (3) et (4) du théorème 2.

En dehors du critère de Dunford-Pettis (qui était d'ailleurs essentiel dans la démonstration du théorème 2) nous disposons donc d'au moins 5 autres critères remarquables de faible compacité relative dans $\mathfrak{M}^1(M)$. Sauf le critère (4) du théorème 2, qui nous a servi surtout d'intermédiaire dans la démonstration, *chacun* de ces critères permet des applications intéressantes et notre article consiste surtout dans l'exposé de ces conséquences. Le §1 exploitait essentiellement le fait que la faible relative compacité de A implique la condition (1) du théorème 2. La réciproque de ce fait implique une réciproque du théorème 1 relatif aux espaces $C(K)$. Aussi consacrerons-nous le reste de ce paragraphe à l'étude d'une propriété réciproque de la propriété D.-P. envisagée dans le paragraphe précédent.

2.2 Une propriété réciproque de la propriété D.-P.

PROPOSITION 8. *Soit E un espace localement convexe, $\mathfrak{S}$ un ensemble de parties convexes cerclées et bornées de E. Les conditions suivantes sont toutes équivalentes:*

(1) *Pour tout espace localement convexe séparé complet F, toute application linéaire continue de E dans F qui transforme les $A \in \mathfrak{S}$ en parties relativement compactes de F, transforme parties bornées de E en parties relativement faiblement compactes de F;*

(2) *Même énoncé, mais F étant supposé un espace de Banach;*

(3) *Toute partie équicontinue A' de E' telle que les $A \in \mathfrak{S}$ soient précompacts pour la A'-convergence, est relativement $\sigma(E', E'')$-compacte;*

ou l'une des formes équivalentes, d'après le lemme 3, *à la condition* (3), *en particulier:*

(4) *Toute partie A' de E', équicontinue, et précompacte pour la topologie de la $\mathfrak{S}$-convergence, est relativement $\sigma(E', E'')$-compacte.*

Démonstration. L'énoncé et la démonstration sont en tous points analogues à ceux de la proposition 1, dont celle-ci est une forme inversée. Il suffit par exemple de prouver (2) → (3), puis (4) → (1).

(2) → (3). On vérifie immédiatement qu'on peut supposer A' convexe, cerclé et faiblement fermé, donc (lemme 6) image de la boule d'un unité dual de Banach F par l'application transposée u' d'une application linéaire continue u de E dans F. Comme u est encore continue quand E est muni de la topologie de la A-convergence (ce qui signifie en effet précisément que u' transforme la boule unité de F' en une partie équicontinue de E'), il suit qu'elle transforme les $A \in \mathfrak{S}$ en parties précompactes de F, donc par hypothèse elle transforme les parties bornées en parties faiblement relativement compactes de F, c'est à dire (lemme 1) u' transforme la boule unité de F' en une partie $\sigma(E', E'')$-compacte de E'.

(4) → (1). Pour montrer que u transforme parties bornées en parties faiblement relativement compactes, il suffit de montrer (lemme 1) que la transposée u' transforme les parties équicontinues de F' en parties relativement $\sigma(E', E'')$-compactes de E', or ces transformées sont équicontinues, et précompactes pour la $\mathfrak{S}$-convergence (lemme 2), la conclusion apparait donc, compte tenu de l'hypothès (4).

DÉFINITION 3. *On dit qu'un espace localement convexe E jouit de la propriété réciproque de la propriété* D.-P. (*en abrégé, que E jouit de la propriété* R. D.-P.) *si les conditions équivalentes* (1) *à* (4) *de la proposition* 8 *sont satisfaites,* $\mathfrak{S}$ *désignant l'ensemble des parties convexes cerclées et faiblement compactes de E.*

Cela signifie donc aussi que toute application linéaire continue de E dans un espace localement convexe complet qui transforme parties faiblement compactes convexes cerclées en parties précompactes, transforme les bornés en parties faiblement relativement compactes. Ou aussi (en vertu de la condition (4)) que les parties équicontinues $\tau(E', E)$-compactes de E' sont $\sigma(E', E'')$-compactes. La conjonction des deux hypothèses D.-P. et R. D.-P. signifie que la condition précédente pour qu'une application linéaire continue de E dans F transforme parties bornées en parties faiblement relativement compactes est à la fois nécessaire et suffisante; ou aussi, que parmi les parties équicontinues de E', relative compacité au sens de $\tau(E', E)$ ou au sens de $\sigma(E', E'')$ est la même chose.

THÉORÈME 4. *Tout espace $C(K)$ (K, espace compact) ou $C_0(M)$ (M, espace localement compact) jouit de la propriété* D.-P. *et* R. D.-P.

Ce n'est autre que le corollaire du théorème 2 du §2.

COROLLAIRE 1. *Soit M un espace normal, $\mathfrak{S}$ un ensemble de parties de M, $C(M, \mathfrak{S})$ l'espace des fonctions continues sur M qui restent bornées sur les ensembles $A \in \mathfrak{S}$, muni de la topologie de la $\mathfrak{S}$-convergence. Alors $C(M, \mathfrak{S})$ jouit de la propriété* D.-P. *et* R. D.-P.

Pour la propriété D.-P. c'est la proposition 6. Pour la propriété R. D.-P., on s'aperçoit, en reprenant le raisonnement qui nous a permis de ramener la proposition 6 au théorème 1, que tout revient à montrer que si A est une partie fermée de M, $C^\infty(A)$ l'espace de Banach des fonctions continues bornées sur A, et (f_i) une suite faiblement convergente dans $C^\infty(A)$, il existe une suite (g_i)

faiblement convergente dans $C(M, \mathfrak{S})$, telle que pour tout i, f_i soit la restriction de g_i à A. *A fortiori* suffit-il de prouver le lemme suivant, qui a son intérêt propre.

LEMME 7. *Soit M un espace normal, A une partie fermée de M, $C^\infty(M)$ et $C^\infty(A)$ les espaces de Banach des fonctions continues et bornées sur M (resp. A). Soit (f_i) une suite de Cauchy faible (resp. une suite faiblement convergente) dans $C^\infty(A)$. Alors il existe une suite de Cauchy faible (resp. une suite faiblement convergente) (g_i) dans $C^\infty(M)$, telle que pour tout i, f_i soit la restriction de g_i à A.*

Pour le voir, remarquons que l'application linéaire de $C^\infty(M)$ dans $C^\infty(A)$ obtenue par restriction à A, applique la boule unité de $C^\infty(M)$ *sur* celle de $C^\infty(A)$ (théorème de Urysohn) de sorte que $C^\infty(A)$ s'identifie à l'espace quotient de $C^\infty(M)$ par le noyau J de cette application, J étant donc l'espace des $f \in C^\infty(M)$ qui s'annulent sur A. Mais on sait que $C^\infty(M)$ s'identifie, en tant qu'algèbre normée complète, à l'espace $C(K)$ des fonctions continues sur le compactifié de Čech-Stone de M, et J s'identifie au sous-espace de $C(K)$ formé des $f \in C(K)$ qui s'annulent sur l'adhérence de A dans K. On est donc ramené au cas où M est un espace compact K. Prenons alors pour tout i un $h_i \in C(K)$ dont la restriction à A soit identique à f_i et considérons l'espace compact métrisable $\tilde{K}$, quotient de K par la relation d'équivalence "$f_i(s) = f_i(t)$ pour tout i." $C(\tilde{K})$ s'identifie alors à un sous-espace de $C(K)$ contenant les h_i, $J \cap C(\tilde{K})$ s'identifiant à l'espace des fonctions continues sur $\tilde{K}$ qui s'annulent sur l'image canonique $\tilde{A}$ de A. On est alors aussitôt ramené à démontrer le lemme 7 quand M est compact métrisable. D'ailleurs, dans le cas où (f_i) est une suite faiblement convergente dans $C(A)$, on peut supposer que la limite est nulle. Notons maintenant que d'après le théorème de Lebesgue, une suite de fonctions continues sur un compact est suite de Cauchy faible (resp. converge faiblement vers zéro) si et seulement si elle est uniformément bornée et converge en chaque point vers une limite (resp. vers une limite nulle). Soit alors (U_i) une *suite* fondamentale de voisinages de A (K est métrisable !) et pour tout i, soit g_i une fonction continue sur K, égale à f_i sur A, à zéro dans $\complement U_i$, et telle que $\|g_i\| = \|f_i\|$ (l'existence en est assurée par le théorème d'Urysohn). Il est alors immédiat de vérifier que la suite (g_i) satisfait aux conditions voulues.

Remarque. La réciproque du théorème 1 n'est pas valable pour les espaces L^1, puisque l'application identique de l'espace l^1 des suites sommables sur lui-même transforme parties faiblement compactes en parties compactes, mais n'est pas faiblement compacte. Comme tout espace L^1 de dimension infini admet un quotient isomorphe à l^1 (ce qui est bien connu, et facile à vérifier), il suit facilement qu'un tel espace ne jouit jamais de la propriété R.D.-P. D'où aussitôt le

COROLLAIRE 2. *Un espace L^1 de dimension infinie ne peut être isomorphe à un quotient*[5] *d'espace $C(K)$*

[5]Signalons pourtant une démonstration directe très facile du corollaire 2. En vertu de la dualité entre les types L^1 et $C(K)$ notée plus haut, il est équivalent à: un espace $C(K)$ de

car on a la

PROPOSITION 9. *Un espace quotient d'un espace de Banach qui jouit de la propriété* R.D.-P., *un facteur direct d'un espace qui jouit de la propriété* R.D.-P., *jouit de la même propriété.—Le produit vectoriel topologique d'une famille d'espaces localement convexes qui jouissent de la propriété* R.D.-P. *jouit de la même propriété.*

La démonstration est triviale.

COROLLAIRE. *Si O est un ouvert de $\mathbf{R}^n$, l'espace $\mathfrak{C}^{(m)}(O)$ des fonctions m fois continûment différentiables sur O, muni de sa topologie usuelle, jouit de la propriété* D.-P. *et* R.D.-P. *Il en est de même de l'espace $\mathfrak{C}^{(m)}(K)$ analogue construit sur une partie compacte K de $\mathbf{R}^n$.*

Il suffit d'appliquer le lemme 4, le théorème 4 et son corollaire 1, et la proposition 9.

2.3 Application de la propriété R.D.-P. Notons d'abord le résultat suivant, qui généralise un théorème de Dunford [**6**]

THÉORÈME 5. *Soit M un espace localement compact muni d'une mesure μ, E un espace localement convexe séparé, Φ une application faiblement sommable de M dans E, telle que $u(f) = \int \Phi f d\mu$ soit élément de E quelle que soit $f \in L^\infty$. Soit $\mathfrak{S}'$ l'ensemble des parties faiblement bornées A' de E' telles que de toute suite extraite de A' on puisse extraire une suite de Cauchy faible. Soit $E(\mathfrak{S}')$ l'espace E muni de la topologie de la $\mathfrak{S}'$-convergence. Alors l'application $f \to u(f)$ de $L^\infty(\mu)$ dans $E(\mathfrak{S}')$ est application précompacte.*

COROLLAIRE. *M et Φ étant comme ci-dessus, l'application $f \to u(f)$ de L^∞ dans E est précompactes dans chacun des cas suivants:*

(a) *E est séparable* (i.e., admet une suite partout dense).
(b) *E est du type* $(\mathfrak{F})$, *Φ est fortement mesurable.*
(c) *E est un espace de Banach réflexif.*
(d) *E est le dual d'un espace F du type* $(\mathfrak{F})$, *muni de la topologie $\tau(F', F)$.*

Démonstration. Soit u' l'application transposée de u. Par définition de l'intégrale faible, pour $x' \in E'$, $u'(x')$ est la classe dans L^1 de la fonction

$$u'(x')\,(t) = \langle \Phi(t), x' \rangle .$$

La conclusion signifie aussi (lemme 2) que pour toute $A' \in \mathfrak{S}'$, $u'(A')$ est partie précompacte de L^1, donc que pour toute suite (x'_i) extraite de A', on peut extraire de la suite $(u(x'_i))$ une suite fortement convergente. Mais par hypothèse, on peut extraire de la suite (x'_i) une suite de Cauchy faible (y'_i); je dis que la

dimension infinie n'est pas isomorphe à un sous-espace vectoriel d'un espace L^1. Mais cela résulte du fait que toute suite de Cauchy faible dans un espace L^1 converge faiblement, tandis qu'il est facile de s'assurer que cette propriété est en défaut dans les espaces $C(K)$ de dimension infinie. — Ce résultat resoud une question de Banach [**1**, pp. 244-245]. Les propriétés 8 et 9 de Banach sont en défaut dans $(\mathfrak{M})$, (m), (C), (C^p).

suite des $u(y'_i)$ est fortement convergente. En effet, u' étant évidemment application faiblement continue de E' dans L^1 (car $\int \Phi f\, d\mu \in E$ pour $f \in L^\infty$), la suite des $u(y'_i)$ est une suite de Cauchy faible dans L^1, donc faiblement convergente [**4**, corollaire du théorème 3], et elle admet d'autre part une suite de représentants

$$u'(y'_i)\,(t) = \langle \Phi(t), y'_i \rangle$$

qui tend en chaque point $t \in M$ vers une limite, d'où résulte (loc. cité, corollaire du théorème 2) que la suite des $u(y'_i)$ converge bien fortement.

Le corollaire en résulte; il suffit en effet de vérifier que de toute suite équicontinue de E' on peut extraire une suite faiblement convergente. C'est vrai sous la condition (a), car les parties équicontinues de E' sont alors faiblement métrisables; sous la condition (b) quand M est "dénombrable à l'infini", parce que l'on peut se ramener à (a) ($\Phi(t)$ prenant ses valeurs presque partout dans un sous-espace séparable de E); sous la condition (c), grâce au théorème de Šmulian appliqué dans le dual; enfin sous la condition (d), grâce au théorème de Šmulian appliqué dans F. Enfin, dans (b), il n'est pas difficile de se libérer de la condition que M soit réunion dénombrable de compacts, car on est ramené à montrer que la restriction de u à $C_0(M)$ est une application pré-compacte (en notant que la boule de $C_0(M)$ est faiblement dense dans celle de $L^\infty(\mu)$), donc que pour toute suite (f_i) extraite de la boule unité de $C_0(M)$, la suite des $u(f_i)$ a une valeur d'adhérence faible; comme chaque f_i est nul dans le complémentaire d'une réunion dénombrable de compacts, on est aussitôt ramené au cas où M est lui-même réunion dénombrable de compacts.

Je ne sais pas s'il est vrai, Φ étant comme dans le théorème 5, que $f \to u(f)$ soit *toujours* une application précompacte de L^∞ dans E. — Dans ce problème, on peut manifestement se ramener au cas où E est un espace de Banach.

PROPOSITION 10. *Soit M un espace localement compact, muni d'une mesure μ, E un espace ($\mathfrak{F}$) jouissant de la propriété* R. D.-P., *Φ une application faiblement sommable de M dans E'. Alors l'application $f \to \int \Phi f\, d\mu$ applique la boule unité de $L^\infty(\mu)$ dans une partie de E' qui est relativement compacte pour $\sigma(E', E'')$ et $\tau(E', E)$.*

En effet, l'image de la boule unité de $L^\infty(\mu)$ est relativement compacte pour $\tau(E', E)$ d'après la partie (d) du corollaire du théorème 5, donc aussi $\sigma(E', E'')$-compacte en vertu de l'hypothèse sur E.

COROLLAIRE 1. *Sous les conditions de la proposition* 10, *si $\mathfrak{F}$ est un filtre borné sur L^∞ qui converge en mesure sur tout compact vers une $g \in L^\infty$, alors $\int \Phi f\, d\mu$ tend fortement suivant $\mathfrak{F}$ vers $\int \Phi g\, d\mu$.*

En effet, l'application v de E dans L^1 définie par

$$v(x)\,(t) = \langle x, \Phi(t) \rangle\,,$$

application qui a pour transposée l'application

$$f \to u(f) = \int \Phi f \, d\mu ,$$

transforme parties bornées en parties faiblement relativement compactes en vertu de la proposition 10 et du lemme 1. Donc (lemme 1) sa bitransposée $v'' = u'$ (application de E'' dans $(L^1)''$ *a priori*) applique E'' dans L^1, de sorte que nous sommes dans les conditions d'application du corollaire 1 de la proposition 3.

Le cas particulier le plus intéressant de ces résultats est énoncé dans le

COROLLAIRE 2. *Soit M un espace localement compact muni d'une mesure μ, K un espace compact, $t \to \nu_t$ une application vaguement sommable de M dans l'espace $\mathfrak{M}^1(K)$ des mesures de Radon sur K. Alors les "mesures composées"*

$$\nu_f = \int f(t)\, \nu_t \, d\mu(t) \;;$$

où f parcourt la boule unité de $L^\infty(\mu)$, admettent une base commune ν, et forment une partie faiblement relativement compacte de l'espace $L^1(\nu)$. Si $\mathfrak{F}$ est un filtre borné dans $L^\infty(\mu)$ qui converge vers $g \in L^\infty(\mu)$ en mesure sur tout compact, alors

$$\int f(t)\, \nu_t \, d\mu(t)$$

converge fortement suivant $\mathfrak{F}$ vers

$$\int g(t)\, \nu_t \, d\mu(t).$$

Il suffit de tenir compte du théorème 5 et de la note du bas de la page 146.

En prenant en particulier pour M un espace discret, avec la masse $+1$ en chaque point, on obtient le

COROLLAIRE 3. *Soit I un ensemble d'indices, E un espace $(\mathfrak{F})$ qui jouit de la propriété* R. D.-P. (*par exemple un espace $C(K)$*), *$(x'_i)_{i \in I}$ une famille d'éléments de E' telle que pour tout $x \in E$, la famille des $\langle x, x'_i \rangle$ soit sommable. Alors la famille des (x'_i) est fortement sommable* (ce qui implique en particulier, si par exemple E est un espace de Banach, que l'ensemble des $i \in I$ tels que $x'_i \neq 0$ est dénombrable).

§ 3. DEUXIÈME CARACTÉRISATION DES APPLICATIONS LINÉAIRES FAIBLEMENT COMPACTES D'ESPACES $C(K)$, ET APPLICATIONS

3.1 La "propriété de Dieudonné". Tout critère de faible compacité relative dans les espaces $\mathfrak{M}^1(K)$ donne par transposition un critère correspondant pour décider qu'une application linéaire d'un espace $C(K)$ est faiblement compacte. Nous laissons au lecteur le soin d'énoncer les divers critères simples, autres que le théorème 4, qu'on peut ainsi tirer du théorème 2 du §2. Le présent paragraphe sera consacré à l'exposé des conséquences du théorème 3 (bien que les résultats les plus essentiels de ce paragraphe pourraient déjà se déduire d'une forme un peu moins fine et bien plus élémentaire du théorème 6 qui va

suivre). Comme d'habitude, nous commençons par une sorte d'axiomatisation des propriétés que nous avons en vue.

PROPOSITION 11. *Soit E un espace localement convexe, Φ un ensemble de filtres dans E qui convergent faiblement dans E'' (resp. $\mathfrak{S}$ un ensemble de parties bornées de E), H le sous-espace vectoriel de E'' engendré par E et les limites des filtres $\mathfrak{F} \in \Phi$ (resp. engendré par E et les adhérences faibles des parties $A \in \mathfrak{S}$). Les conditions suivantes sont toutes équivalentes:*

(1) *Pour tout espace localement convexe séparé complet F, toute application linéaire continue de E dans F qui transforme les filtres $\mathfrak{F} \in \Phi$ en filtres qui convergent faiblement dans F (resp. qui transforme les parties $A \in \mathfrak{S}$ en parties relativement faiblement compactes de F), transforme les parties bornées de E en parties faiblement relativement compactes de F;*

(1) bis. *Pour tout espace localement convexe séparé complet F, et toute application linéaire continue u de E dans F telle que sa bitransposée u'' applique H dans F, u'' applique aussi E'' dans F (i.e., u transforme parties bornées de E en parties faiblement relativement compactes de F —voir lemme* 1*);*

(2) *Comme* (1), *mais F étant astreint à être un espace de Banach;*

(2) bis. *Comme* (1) bis, *mais F étant astreint à être un espace de Banach;*

(3) *Toute partie équicontinue convexe cerclée et $\sigma(E', H)$-compacte de E' est aussi $\sigma(E', E'')$-compacte.*

Démonstration. Il suffit de démontrer l'énoncé relatif à l'ensemble Φ de filtres, celui relatif à la donnée de $\mathfrak{S}$ en est un cas particulier comme on voit en considérant l'ensemble Φ des filtres de Cauchy faibles sur les $A \in \mathfrak{S}$. On a évidemment (1) $\rightarrow$ (2), et (1) bis et (2) bis sont respectivement équivalents à (1) et (2), comme il résulte aussitôt du fait que u'' s'obtient en prolongeant u par continuité faible. On est donc ramené à montrer (2) bis $\rightarrow$ (3) et (3) $\rightarrow$ (1).

(2) bis $\rightarrow$ (3). Soit en effet A' une partie équicontinue convexe cerclée et $\sigma(E', H)$-compacte de E', en vertu du lemme 6 du §2, c'est l'image de la boule unité d'un dual d'espace de Banach F, par l'application transposée u' d'une application linéaire continue u de E dans F. Pour montrer que A' est $\sigma(E', E'')$-compact, il suffit donc (lemme 1) de montrer que la bitransposée u'' applique E'' dans F, et par hypothèse il suffit même de montrer que u'' applique H dans F. Cela se démontre comme dans le lemme 1: E est dense dans H pour la topologie $\tau(H, E')$ (car toute forme linéaire sur H continue pour cette topologie —i.e. provenant d'un $x' \in E'$— et qui s'annule sur E, est nulle), or la restriction de u'' à H est continue pour la topologie $\tau(H, E')$ (et la topologie naturelle sur F''), d'après l'hypothèse sur A'; comme u'' applique E dans F et que F est complet, la conclusion apparait.

(3) $\rightarrow$ (1). Soit u une application linéaire continue de E dans F telle que $u''(H) \subset F$, nous voulons montrer, moyennant (3), que u'' transforme parties bornées de E en parties faiblement relativement compactes de F, i.e. (lemme 1) que u' transforme parties équicontinues convexes cerclées et faiblement compactes B' de F' en parties $\sigma(E', E'')$-compactes de E'. Mais $u''(H) \subset F$ signifie

aussi que u' est continue de F' faible dans E' muni de $\sigma(E', H)$, de sorte que les $u'(B')$ sont $\sigma(E', H)$-compactes, donc $\sigma(E', E'')$-compactes en vertu de (3).

DÉFINITION 4. *Soit E un espace localement convexe. On appelle sous-espace de Baire de classe 1 de E'', le sous-espace de E'' formé des limites faibles dans E'' des suites de Cauchy faibles de E. On dit que E jouit de la propriété de Dieudonné (en abrégé, de la propriété D) s'il satisfait aux conditions équivalentes de la proposition* 11, *Φ étant l'ensemble des filtres élémentaires associé aux suites de Cauchy faibles dans E, donc H le sous-espace de Baire de classe* 1 *de E''.*

Cela signifie donc que toute application linéaire continue de E dans un espace localement convexe séparé complet F, qui transforme suites de Cauchy faibles de E en suites faiblement convergentes de F, transforme les parties bornées de E en parties faiblement relativement compactes de F.

On pourrait aussi, pour tout nombre ordinal α de seconde classe, introduire de façon évidente le "sous-espace de Baire de classe α" de E'', et la "propriété D de classe α" correspondante; et aussi le "sous-espace de Baire" de E'', réunion des sous-spaces de Baire de toutes les classes, et une "propriété D au sens large" correspondante. Cette dernière propriété serait suffisante pour la plupart des applications que nous avons en vue. Mais comme dans les espaces que nous allons considérer, nous avons même la propriété plus stricte de la définition 4, ce sera la seule dont nous parlerons.

Exemples. (a) Tout espace semi-réflexif jouit trivialement de la propriété D.

(b) Si E est complet et si dans E toute suite de Cauchy faible converge faiblement, alors E ne jouit de la propriété D que s'il est semi-réflexif, car l'application identique de E sur lui-même doit transformer parties bornées en parties faiblement relativement compactes. [**3,** p. 79]. — En particulier, un sous-espace vectoriel fermé d'un espace du type L^1 ne jouit de la propriété D que s'il est réflexif.

(c) Si les parties bornées de E sont faiblement métrisables (ce qui signifie, lorsque E est un espace de Banach, que son dual fort est séparable), E jouit de la propriété D, car le sous-espace de Baire de classe 1 de E'' est alors identique à E''.

(d) L'exemple le plus important fera l'objet du théorème 6.

PROPOSITION 12. *Un espace quotient d'un espace de Banach qui jouit de la propriété D, tout facteur direct d'un espace qui jouit de la propriété D, jouit de la même propriété. Le produit vectoriel topologique d'une famille d'espaces qui jouissant chacun de la propriété D, jouit de la propriété D.*

Soit E un espace localement convexe qui jouit de la propriété D, F un espace localement convexe séparé complet tel que toute suite de Cauchy faible dans F converge faiblement dans F. Alors toute application linéaire continue de E dans F transforme parties bornées en parties faiblement relativement compactes de F.— Il en est en particulier ainsi chaque fois que F est un espace du type L^1 [**4,** cor. du théorème 3].

La démonstration est triviale. La deuxième partie de la proposition constitue l'application la plus importante de la propriété D (voir théorèmes 7, 7 bis, 8).

3.2 Cas des espaces $C(K)$. Soit M un espace localement compact, $\mathfrak{L}^\infty(M)$ l'espace des fonctions complexes bornées sur M qui sont mesurables pour toute mesure de Radon sur M. Comme nous l'avons déjà signalé, $\mathfrak{L}^\infty(M)$ s'identifie à un sous-espace normé du bidual de l'espace $E = C_0(M)$ des fonctions complexes continues sur M nulles à l'infini. Alors le sous-espace de Baire de classe 1 de E'' s'identifie à l'espace des fonctions de Baire bornées de classe 1 au sens classique (limites des suites convergentes uniformément bornées de fonctions continues à support compact), car d'après le théorème de Lebèsgue, la suite (f_i) extraite de $\mathfrak{L}^\infty(M)$ est une suite de Cauchy faible si (et seulement si) elle est uniformément bornée, et converge vers une limite en chaque point $t \in M$.
Un des résultats les plus importants de ce travail est le

THÉORÈME 6. *Soit K un espace compact. Alors l'espace $C(K)$ jouit de la propriété D. De facon plus précise, si u est une application linéaire continue de $C(K)$ dans un espace localement convexe séparé complet F, les conditions suivantes sont toutes équivalentes:*

(1) *u est application faiblement compacte,*

(2) *Pour tout partie fermée A de K — soit ϕ_A sa fonction caractéristique, identifiée à un élément du bidual de $C(K)$ — on a $u''(\phi_A) \in F$. Dans cet énoncé, on peut aussi supposer que A est astreint à admettre un système fondamental dénombrable de voisinages.*

(3) *u transforme toute suite croissante de fonctions continues positives majorées par* 1 *en une suite faiblement convergente dans F.*

Démonstration. (1) implique évidemment (3), car une suite croissante de fonctions continues positives majorées par 1 est uniformément bornée et converge en chaque point, donc est une suite de Cauchy faible dans $C(K)$. Par ailleurs, la forme la plus faible de (2) est manifestement équivalente à l'énoncé analogue qu'on obtient en supposant que A est un ouvert réunion d'une suite de compacts. Mais il est facile de vérifier, grâce au théorème d'Urysohn, qu'alors ϕ_A est limite d'une suite croissante de fonctions continues positives majorées par 1, d'où suit que (3) implique (2). Nous sommes donc ramenés à montrer que les conditions de la proposition 11 sont vérifiées quand H est le sous-espace de E'' engendré par E et les fonctions caractéristiques des parties fermées de K ayant une *suite* fondamentale de voisinages. Mais en vertu de la condition (3) de la proposition 11, ce n'est autre que la forme renforcée du théorème 3 du §2, dans la remarque qui suit ce théorème.

Remarque 1. Notons à titre didactique qu'une démonstration bien plus élémentaire permet de démontrer la forme un peu moins fine du théorème 6: Si u'' applique l'espace des fonctions de Baire bornées de classe 2 *dans F*, alors u est application faiblement compacte (i.e., $C(K)$ possède la "propriété D de classe 2"). Grâce au lemme 5, on se ramène d'abord au cas où K est métrique,

donc $C(K)$ séparable, et on note que les conditions de la proposition 11 sont équivalentes, quand E est séparable, à la condition:

(4) *Toute suite équicontinue dans E' qui converge vers zéro au sens de la topologie $\sigma(E', H)$, converge vers zéro au sens de $\sigma(E', E'')$.* — Mais dans le cas actuel, toute suite dans $\mathfrak{M}^1(K)$ est contenue dans un espace $L^1(\mu)$. On se sert alors du fait que le dual de $L^1(\mu)$ est $L^\infty(\mu)$, et que toute $f \in L^\infty(\mu)$ a une fonction-représentant qui est une fonction de Baire de classe 2.

Remarque 2. Le lecteur vérifiera sans difficulté à l'aide des techniques courantes dans cet article, que l'énoncé du théorème 6 est textuellement valable pour les espaces $C_0(M)$, où M est un espace localement compact. De même, le raisonnement qui a permis de déduire la proposition 6 du théorème 1, permet aussi de montrer que, si M est un espace localement compact, alors l'espace $C(M)$ des fonctions complexes continues sur M, muni de la topologie de la convergence compacte, jouit de la propriété D, et satisfait aussi à un énoncé analogue au théorème 6 (on utilise le lemme 7). Nous énoncerons donc, compte tenu de la proposition 12:

COROLLAIRE 1 DU THÉORÈME 6. *Tout produit topologique d'espaces vectoriels du type $C(M)$, tout espace quotient d'un espace du type $C(K)$ ou du produit topologique d'un nombre fini d'espaces du type $C(K)$, tout espace facteur direct d'un produit topologique d'espaces du type $C(M)$—jouit de la propriété D. En particulier, les espaces $\mathfrak{E}^{(m)}(O)$ et $\mathfrak{E}^{(m)}(K)$ de L. Schwartz* (voir proposition 7 et lemme 4) *ont la propriété D.*

COROLLAIRE 2. *Un espace de Banach qui est à la fois isomorphe à un espace quotient d'un espace $C(K)$ et à sous-espace d'un espace L^1 est forcément réflexif.*

Car nous avons déjà remarqué qu'un sous-espace vectoriel fermé d'un espace L^1 ne jouit de la propriété D que s'il est réflexif.—Ce corollaire précise de beaucoup le corollaire 2 analogue du théorème 4.

3.3 Application aux formes bilinéaires sur les produits d'espaces du type $C(K)$. La deuxième partie de la proposition 12 donne l'application la plus importante du théorème 6 et de ses corollaires. En particulier, on a le résultat suivant, qui mérite mention explicite.

THÉORÈME 7. *Toute application linéaire continue d'un espace $C(K)$ dans un espace L^1 est faiblement compacte, et transforme parties faiblement compactes ou parties bornées faiblement métrisables de $C(K)$ en parties fortement relativement compactes, donc les suites de Cauchy faibles en suites fortement convergentes.*

(La deuxième partie du théorème résulte du théorème 1, et de la remarque qui suit la proposition 1 bis.)

Rappelons maintenant que le dual d'un espace $C(K)$, donc l'espace $\mathfrak{M}^1(K)$ des mesures de Radon sur K, est isomorphe à un espace L^1 (voir remarques suivant le corollaire du théorème 1), d'autre part il est bien connu que pour

deux espaces de Banach E et F donnés, les formes bilinéaires continues sur $E \times F$ correspondent biunivoquement aux applications linéaires continues de E dans F'. Posons maintenant la

DÉFINTION 5. *Une forme bilinéaire continue sur le produit de deux espaces de Banach E et F est dite compacte (resp. faiblement compacte), si l'application linéaire correspondante de E dans F' fort est compacte (resp. faiblement compacte).*

On vérifie d'ailleurs aussitôt qu'il revient au même de dire que l'application linéaire *de F dans E'* fort qui correspond à la forme bilinéaire est compacte (resp. faiblement compacte) car chacune des deux applications linéaires est la restriction (à E resp. à F) de la transposée de l'autre. Cela étant, on a le

THÉORÈME 7 BIS. *Soient E et F deux espaces de Banach isomorphes respectivement à des quotients d'espaces du type $C(K)$. Alors*

(a) *Toute forme bilinéaire continue sur $E \times F$ est faiblement compacte.*

(b) *Soit G un espace localement convexe, u une application bilinéaire continue de $E \times F$ dans G. Si on suppose que E est même isomorphe à un espace du type $C(K)$, alors, A désignant une partie faiblement compacte ou bornée et faiblement métrisable de E, et V la boule unité de F, u est fonction uniformément continue sur $A \times V$, muni de la structure uniforme produit des structures uniformes faibles, dans G faible.*

Le première partie du théorème résulte des remarques qui précédaient et proposition 12. Pour la deuxième partie, il faut vérifier que pour toute forme linéaire continue z' sur G, la fonction $\langle u(x, y), z'\rangle$ est uniformément continue sur $A \times V$, ce qui nous ramène au cas où G est le corps des scalaires. Mais alors, on vérifie trivialement que cet énoncé est équivalent à l'énoncé du théorème 7.

Le théorème 7 redonnerait par une autre voie le corollaire 2 de la proposition 10 du §2, en utilisant le fait que $L^\infty(\mu)$ est un espace du type $C(K)$ (Gelfand-Stone) et $\mathfrak{M}^1(K)$ un espace du type L^1 (Kakutani). Nous y reviendrons au No. 5 de ce paragraphe.

3.4 Application à une classe remarquable d'applications linéaires.

Définition 6. Soient E et F deux espaces localement convexes. Une forme bilinéaire u sur $E \times F$ est dite "intégrale", si on peut trouver une partie équicontinue faiblement fermée A' (resp. B') de E' (resp. F') et une mesure de Radon μ sur le produit topologique $A' \times B'$ des compacts faibles A' et B', de telle façon que l'on ait, pour $x \in E$ et $y \in F$

3.41 $$u(x, y) = \int_{A' \times B'} \langle x, x'\rangle\langle y, y'\rangle \, d\mu(x', y').$$

G étant un autre espace localement convexe, une application linéaire v de E dans G est dite intégrale si la forme bilinéaire $\langle vx, y'\rangle$ sur $E \times$ (G' fort) est intégrale, c'est à dire s'il existe une partie équicontinue faiblement fermée A' de E' et une partie bornée B dans G (dont nous désignons l'adhérence faible dans G'' par $\tilde{B}$)

et enfin une mesure μ sur l'espace produit $A' \times \bar{B}$ des espaces compacts faibles A' et $\bar{B}$, tels que l'on ait, pour $x \in E$, $y' \in G$:

3.42 $$\langle v(x), y' \rangle = \int_{A' \times \bar{B}} \langle x, x' \rangle \langle y, y' \rangle \, d\mu(x', y).$$

La relation (3.41) s'écrit sous forme abrégée

$$u = \int_{A' \times B'} x' \otimes y' \, d\mu(x', y')$$

où pour $x' \in E'$ et $y' \in F'$, $x' \otimes y'$ désigne la forme bilinéaire élémentaire $(x, y) \to \langle x, x' \rangle \langle y, y' \rangle$ sur $E \times F$, le signe $\int$ désignant une intégrale faible dans l'espace de toutes les formes bilinéaires sur $E \times F$, mis en dualité avec l'espace $E \otimes F$ (voir [**2**]).

On peut supposer, pour étudier une forme bilinéaire intégrale ou une application intégrale, que l'on ait avec les notations précédentes: $\|\mu\| < 1$, A' et B' (resp. A' et B) sont convexes cerclés. La formule de la moyenne donne alors aussitôt que $x \in (A')^\circ$, $y \in (B')^\circ$ implique $|u(x, y)| < 1$, par suite, u est application bilinéaire continue sur $E \times F$. Cela signifie aussi que l'application linéaire v de E dans F' définie par $\langle y, vx \rangle = u(x, y)$ applique le voisinage $(A')^\circ$ de l'origine dans E dans la partie équicontinue B' de F', *a fortiori* u est continue de E dans F' fort. Cela implique enfin qu'une application linéaire intégrale de E dans G, donné par la formule (3.42), applique le voisinage $(A')^\circ$ de O dans E dans la partie bornée B de G.

Les application linéaires intégrales interviennent assez souvent en pratique. D'ailleurs, elles admettent une interprétation fonctionnelle très simple, que je me borne à signaler ici sans démonstration: Les formes bilinéaires intégrales sur $E \times F$ s'identifient aux formes linéaires continues sur l'espace $E \otimes F$ muni de la topologie de la convergence uniforme sur les produits de parties équicontinues.[6] La propriété que nous avons en vue ici est donnée par le

THÉORÈME 8. *Une application linéaire intégrale v d'un espace localement convexe E dans un espace localement convexe séparé complet G est faiblement compacte, et transforme parties faiblement compactes ou parties bornées et faiblement métrisables en parties relativement compactes, suites de Cauchy faibles en suites fortement convergentes. Par suite, une application linéaire de E dans G qui s'obtient en composant une application linéaire intégrale de E dans un espace localement convexe séparé complet F et une application linéaire intégrale de F dans G, est compacte.*

Démonstration. Munissons comme d'habitude le bidual G'' de G de la topologie T de la convergence uniforme sur les parties équicontinues de G': topologie qui induit sur G sa topologie propre. Posons $F = G'$ fort. Je dis que v

[6] *Note ajoutée pendant la correction des épreuves.* La théorie des applications linéaires intégrales est exposée systématiquement dans mon travail *Produits tensoriels topologiques et espaces nucléaires* (à paraître dans les Memoirs of the American Mathematical Society), Chap. 1, §4, no. 3. J'y obtiens des résultats plus précis que le théorème 8 et son corollaire.

transforme $(A')^\circ$ en une partie faiblement relativement compacte de G. Il suffit de montrer que c'est une partie de F' qui est relativement compacte pour la topologie $\sigma(F', F'')$, car *a fortiori* le sera-t-elle pour la topologie faible qui correspond à la topologie T sur F', or elle est contenue dans le sous-espace G de F' qui est fermé pour T (puisque complet). De même, pour montrer que v transforme parties faiblement compactes et parties bornées faiblement métrisables de E en parties relativement compactes de G, il suffit de montrer que ces dernières sont compactes dans F' fort. (L'énoncé relatif aux suites de Cauchy faibles en sera un corollaire immédiat.) Nous sommes par suite ramenés à la situation de la première partie de la définition 6. Nous supposerons alors A' et B' convexes cerclés. Soit E_1 l'espace de Banach obtenu en séparant et complétant E pour la semi-norme définie par le voisinage $(A')^\circ = U$ de l'origine dans E, et F_1 l'espace de Banach analogue déduit de F à l'aide du voisinage $(B')^\circ = V$ de l'origine. Nous avons déjà remarqué (lemme 6) que les boules unité des duals respectifs de E_1 et F_1 s'identifient respectivement, avec leur topologie faible, à A' faible et B' faible, par suite la mesure μ définit aussi une forme bilinéaire intégrale $\tilde{u}$ sur $E_1 \times F_1$, d'où une application linéaire intégrale $\tilde{v}$ de E_1 dans F_1'. Cela ramène le théorème envisagé au cas où E et F sont des espaces de Banach, car il suivra que v applique U dans une partie de $\mathbf{C}.B'$ (espace engendré par B') qui est faiblement relativement compacte dans l'espace $\mathbf{C}.B'$ muni de la topologie définie par la boule B', donc *a fortiori* relativement compact dans F' pour $\sigma(F', F'')$, et remarques analogues pour les autres parties de l'énoncé du théorème.

Supposons donc que E et F sont des espaces de Banach, A' et B' les boules unité faibles de leur duals respectifs. Alors E et F s'identifient respectivement à des sous-espaces normés des espaces $C(A')$ et $C(B')$ et la forme bilinéaire u donnée par 3.41 se prolonge de façon naturelle en une forme bilinéaire continue $\tilde{u}$ sur $C(A') \times C(B')$, donnée par

$$\tilde{u}(f, g) = \int_{A' \times B'} f(x')g(y')\, d\mu(x', y').$$

L'application linéaire v de E dans F' se déduit de l'application linéaire de $C(A')$ dans le dual $\mathfrak{M}^1(B')$ de $C(B')$ déduite de la forme bilinéaire $\tilde{u}$, en prenant la restriction de $\tilde{v}$ à E, et en composant avec l'application naturelle de $\mathfrak{M}^1(B')$ sur F'. L'application du théorème 7 donne alors les diverses propriétés que nous voulions établir.

Corollaire 1. *Une application linéaire intégrale d'un espace de Banach E dans un autre F est compacte lorsque l'un des deux espaces est réflexif.*

Cela résulte aussitôt du théorème 8 lorsque le premier espace est réflexif, dans le cas contraire il suffit de considérer la transposée.

Corollaire 2. *Si u est forme bilinéaire intégrale sur $E \times F$, sa restriction au produits de deux parties de E et F qui sont soit faiblement compactes, soit faiblement métrisables et bornées, est continue pour le produit des topologies faibles.*

Cela résulte du théorème 8, comme le théorème 7 bis du théorème 7. Bien entendu, on pourrait même énoncer un résultat plus fort, analogue à l'énoncé du théorème 7 bis.

Remarque. En fait, nous avons démontré mieux que le théorème 8, savoir: Si v est une application de E dans l'espace localement convexe complet G donnée par la formule 3.42, où B est une partie bornée convexe cerclée et fermée de G, et si on désigne par G_1 l'espace de Banach $\mathbf{C}.B$, muni[7] de la norme définie par la "boule" B, alors v transforme le voisinage $(A')^\circ$ de l'origine dans E en une partie relativement faiblement compacte de G_1, et les parties faiblement compactes ou faiblement métrisables et bornées de E en parties relativement compactes de G_1. Cet énoncé est notablement plus fort que celui du théorème 8; il joue un rôle important dans la théorie des "espaces nucléaires" annoncée dans [**10**].

3.5 Deux autres applications du théorème 6.

PROPOSITION 13. *Soit M un espace localement compact muni d'une mesure μ, F un espace localement convexe séparé complet, Φ une application faiblement sommable de M dans F. Pour qu'on ait $\int \Phi f \, d\mu \in F$ pour toute $f \in L^\infty(\mu)$, il faut et il suffit qu'on ait*

$$\int_A \Phi f \, d\mu \in F$$

pour toute partie fermée A de M.

Supposons pour simplifier que F est un espace de Banach (la démonstration est essentiellement la même dans le cas général). *A priori*, pour $f \in L^\infty(\mu)$, $\int \Phi f \, d\mu$ est un élément du bidual F'' de F, et l'application

$$f \to \int \Phi f \, d\mu = u(f)$$

est une application linéaire de L^∞ dans E'', continue pour les topologies faibles des duals par définition même de l'intégrale faible, donc fortement continue d'après le théorème du "graphe fermé". Mais la boule unité de $C_0(M)$ définit une partie faiblement dense de la boule de L^∞, donc pour montrer que $u(f) \in F$ pour toute $f \in L^\infty$, il suffit de montrer que l'ensemble des $\int \Phi f \, d\mu$, quand f parcourt le boule unité de $C_0(M)$, est partie faiblement relativement compacte de F. Comme u est fortement continue de L^∞ dans F'', il résulte facilement de l'hypothèse que u applique les éléments de L^∞ qui proviennent d'une $f \in C_0(M)$ *dans* F (car toute $f \in C_0(M)$ peut s'approcher uniformément par des combinaisons linéaires de fonctions caractéristiques d'ensembles compacts, comme on vérifie facilement). Par suite, $f \to \int \Phi f \, d\mu$ est une application linéaire continue v de $C_0(M)$ *dans* F, nous voulons montrer qu'elle est faiblement compacte.

[7]Il bien s'agit d'un espace *complet*, car on démontre facilement: Si A est une partie convexe cerclée et complète d'un espace localement convexe, alors l'espace $\mathbf{C}.A$, muni de la norme définie par la "boule" A, est *complet*.

D'après le théorème 6 (remarque 2) il suffit de montrer que pour toute partie fermée A de M, on a $v''(\phi_A) \in F$. Mais on vérifie trivialement que l'on a $v''(\phi_A) = u(\phi_A)$, donc par hypothèse $v''(\phi_A) \in F$, cqfd.

On fera attention qu'il n'est nullement suffisant que l'on ait $\int \Phi f\,d\mu \in F$ pour toute $f \in C_0(M)$ pour avoir la même relation pour toute $f \in L^\infty(\mu)$, même dans le cas où M est l'intervalle compact (0,1) muni de la mesure de Lebèsgue, et où Φ est fortement mesurable.

Une dernière application du théorème 6 est relative à la notion de fonction vectorielle complètement additive d'ensemble borélien,[8] définie sur la tribu borélienne T attachée à un espace compact K, et à valeurs dans un espace localement convexe E. Quand E est le corps des scalaires il est bien connu (théorème de Riesz) que cette notion est identique à celle de mesure de Radon sur K, c'est à dire de forme linéaire continue sur l'espace $C(K)$. D'autre part, il est connu aussi que, E étant de nouveau quelconque, une fonction *faiblement* complètement additive sur T, à valeurs dans E, est déjà *fortement* complètement additive: c'est l'essentiel du théorème d'Orlicz: une famille (x_i) d'éléments de E telle que toute sous-famille soit faiblement sommable, est fortement sommable (le plus souvent, on l'énonce seulement quand E est une espace de Banach, mais la démonstration est générale—c'est par exemple un cas particulier du corollaire 2 de la proposition 3).

A partir du théorème de Riesz, il est immédiat de voir, par transposition, que la notion de fonction vectorielle complètement additive d'ensemble borélien, à valeurs dans E, est équivalente à la notion d'application linéaire u de E' dans le dual $\mathfrak{M}^1(K)$ de $C(K)$, telle que pour tout $A \in T$, la forme linéaire

$$x' \to \langle \phi_A, u(x') \rangle$$

sur E' provienne d'un élément de E (ϕ_A désignant la fonction caractéristique de A, identifiée à une forme linéaire sur $\mathfrak{M}^1(K)$), c'est à dire dont la transposée u' (qui *a priori* est une application linéaire du dual de $\mathfrak{M}^1(K)$ dans le dual *algébrique* de E') applique *dans* E le sous-espace β du dual de $\mathfrak{M}^1(K)$ engendré par les fonctions caractéristiques d'ensembles boréliens. Une telle application u transforme une partie convexe cerclée équicontinue et faiblement compacte A' de E' en une partie *bornée* de $\mathfrak{M}^1(K)$. Cela résulte du "théorème du graphe fermé" de Banach quand E est un espace de Banach, et on se ramène à ce cas dans le cas général, en considérant u comme une application linéaire de l'espace de Banach $\mathbf{C}A'$ (muni de la "boule" A') dans $\mathfrak{M}^1(K)$. Soit alors E'^* l'espace des formes linéaires sur E' qui sont bornées sur les parties équicontinues de E', muni de la topologie de la convergence uniforme sur les parties équicontinues (topologie qui induit sur E la topologie propre de E). De ce qui précède, il

[8] Pour abréger, nous appellerons, contrairement à l'usage courant, ensemble borélien toute partie de K dont la fonction caractéristique est une fonction de Baire, c'est à dire appartient à la plus petite famille de fonctions sur K contenant les fonctions continues, et stable pour les limites de suites qui convergent en chaque point. Cette terminologie coïncide avec la terminologie classique si K est métrisable.

résulte que u' est application linéaire *continue* du bidual de $C(K)$ dans E'^*. Si E est complet, u' applique dans E l'adhérence forte $\bar{\beta}$ de β dans le dual de $\mathfrak{M}^1(K)$, c'est à dire l'espace $\mathfrak{B}(K)$ des fonctions de Baire sur K. En particulier, la restriction de u' à $C(K)$ sera une application linéaire continue v de $C(K)$ dans E, et on vérifie aussitôt que u n'est autre que la transposée de v, dont la bitransposée v'' est donc égale à u' et applique $\mathfrak{B}(K)$ dans E. Réciproquement, la transposée $u = v'$ d'une telle application linéaire continue v de $C(K)$ dans E satisfait évidemment à la condition envisagée plus haut, savoir que u' applique β dans E. Ainsi, il résulte élémentairement du théorème de Riesz que *les fonctions complètement additives d'ensembles boréliens de K, à valeurs dans l'espace localement convexe séparé complet E, correspondent biunivoquement aux applications linéaires continues v de $C(K)$ dans E dont la bitransposée transforme les fonctions de Baire en éléments de E.* Compte tenu alors de la condition (2) du théorème 6, on obtient:

PROPOSITION 14. *Soit K un espace compact, E un espace localement convexe séparé complet. Alors les fonctions vectorielles d'ensembles boréliens, à valeurs dans E, complètement additives (faiblement ou fortement) correspondent biunivoquement aux applications linéaires faiblement compactes de $C(K)$ dans E.*

Bien entendu, on obtient la fonction d'ensemble définie par l'application linéaire v, en prenant la restriction de la bitransposée de v aux fonctions caractéristiques d'ensembles boréliens.

§4. SUR DEUX CLASSES PARTICULIÈRES D'ESPACES $C(K)$

4.1 K est un espace compact "stonien". Un espace compact est dit "stonien", si l'adhérence d'une partie ouverte est ouverte; il revient au même de dire que l'espace des fonctions réelles continues sur K est un lattice complet pour sa relation d'ordre naturelle. Les espaces stoniens interviennent actuellement dans de nombreuses questions. En particulier, si on identifie, comme nous l'avons déjà fait plusiers fois, un espace $L^\infty(\mu)$ à un espace $C(K)$, K sera un espace stonien (car la partie réelle de $L^\infty(\mu)$ est un lattice complet, comme il est bien connu). Signalons encore le résultat suivant, du à L. Nachbin [**12**]: Si K est un espace compact stonien, et si $C(K)$ est un sous-espace vectoriel normé d'un espace de Banach E, alors il existe une projection de E sur $C(K)$ de norme égale à 1.

Les espaces stoniens les plus simples sont les compactifiés de Čech-Stone $\hat{I}$ d'espaces discrets arbitraires I. $C(\hat{I})$ s'identifie alors à l'espace $C^\infty(I)$ de toutes les fonctions complexes et bornées sur I, et (par l'intermédiaire de leurs fonctions caractéristiques) les parties arbitraires de I correspondent biunivoquement aux parties à la fois ouvertes et fermées de $\hat{I}$, avec conservation de la relation d'inclusion. Il est facile de plus de vérifier le fait bien connu que les formes linéaires continues sur $C^\infty(I) = C(\hat{I})$, c'est à dire les mesures de Radon sur $\hat{I}$, correspondent biunivoquement aux fonctions complexes additives définies sur l'ensemble de *toutes* les parties de I.

Tout ceci posé, nous pouvons démontrer le

THÉORÈME 9. *Soit K un espace compact Stonien. Alors toute suite (μ_i) de mesures de Radon sur K, vaguement convergente vers zéro, converge vers zéro pour la topologie faible de l'espace de Banach $\mathfrak{M}^1(K)$.*

Démonstration. $C(K)$ peut s'identifier à un sous-espace vectoriel normé de l'espace E de *toutes* les fonctions bornées définies sur K, et de plus c'est un facteur direct de cet espace d'après le théorème de Nachbin rappelé plus haut. Donc la suite des formes linéaires μ_i sur $C(K)$ peut se prolonger en une suite de formes linéaires continues ν_i sur E, telle que

$$\lim_i \nu_i(f) = 0$$

pour toute $f \in E$. Il suffit maintenant de montrer que (ν_i) converge vers zéro pour $\sigma(E', E'')$, ce qui nous ramène au cas où l'espace compact envisagé est le compactifié de Čech-Stone $\hat{I}$ d'un espace discret I.—Tout revient manifestement à montrer que la suite (μ_i) est partie faiblement relativement compacte de l'espace de Banach $\mathfrak{M}^1(\hat{I})$. Nous appliquons pour cela le critère (3) du théorème 2, en notant qu'ici ce critère signifie manifestement qu'on ne peut trouver un $\epsilon > 0$, une suite partielle (ν_i) de (μ_i), et une suite (O_i) d'ouverts deux à deux disjoints, tels qu'on ait $|\nu_i(O_i)| > \epsilon$ pour tout i. D'ailleurs, comme l'espace $\hat{I}$ est totalement discontinu, on pourrait alors pour tout i trouver une partie O'_i de O_i *à la fois ouverte et fermée*, telle que l'on ait encore $|\nu_i(O'_i)| > \epsilon$, de sorte qu'on pourrait déjà supposer les O_i à la fois ouverts et fermés. Interprètons maintenant les ν_i comme fonctions additives dans l'ensemble de toutes les parties de I, et identifions les O_i à des parties deux à deux disjointes A_i de I, on aurait donc $|\nu_i(A_i)| > \epsilon$ pour tout i, montrons que cela est impossible. En effet, soit N l'ensemble des entiers naturels, et pour tout i soit $\bar{\nu}_i$ la fonction additive d'ensembles, définie sur l'ensemble de toutes les parties B de N par la formule

$$\bar{\nu}_i(B) = \nu_i(\bigcup_{j \in B} A_j).$$

Les normes des ν_i, considérés comme formes linéaires sur $C^\infty(N)$, restent bornées, et on a

$$\lim_i \bar{\nu}_i(B) = 0$$

pour tout $B \subset N$. Il en résulte, d'après un lemme du à R. Philipps [**13**], que l'on a

$$\lim_i \sum_j |\bar{\nu}_i(j)| = 0,$$

a fortiori $\lim \bar{\nu}_i(i) = 0$, i.e., $\lim \nu_i(A_i) = 0$, ce qui prouve le théorème.

COROLLAIRE 1. *Si K est un espace compact stonien, F un espace localement convexe séparé complet séparable* (i.e. admettant une suite partout dense), *alors toute application linéaire continue de $C(K)$ dans F est faiblement compacte.*

En effet, cela est inclus dans le

LEMME 8. *Si E est un espace de Banach, les deux conditions suivantes sont équivalentes:*

(1) *Toute suite (x'_i) du dual E', faiblement convergente vers zéro, converge vers zéro pour $\sigma(E', E'')$.*

(2) *Toute application linéaire continue u de E dans un espace localement convexe séparé complet séparable F, est faiblement compacte.*

(1) $\rightarrow$ (2), car il suffit de montrer (lemme 1) que la transposée u' transforme parties équicontinues de F' en parties de E' qui sont relativement $\sigma(E', E'')$-compactes. Mais les parties équicontinues faiblement fermées de F' sont faiblement compactes et *métrisables*, donc leurs images dans E' ont la même propriété. Or, une partie A' bornée et faiblement métrisable dans E' est effectivement relativement $\sigma(E', E'')$-compacte moyennant (1). Car en vertu du théorème d'Eberlein, il suffit de montrer que de toute suite extraite de A' on peut extraire une suite qui converge pour $\sigma(E', E'')$, or on peut en effet en extraire une suite qui soit *faiblement* convergente, donc aussi $\sigma(E',E'')$-convergente par hypothèse.

(2) $\rightarrow$ (1), car soit (x'_i) une suite dans E' qui converge vers zéro faiblement, il suffit de montrer qu'elle est relativement compacte pour $\sigma(E', E'')$. Mais considérons l'application linéaire u de E dans l'espace c_0 des suites de nombre complexes qui tendent vers zéro, définie par $u(x) = (\langle x, x'_i \rangle)$. Sa transposée transforme la boule unité du dual l^1 de c_0 en une partie de E' contenant les x'_i. D'après l'hypothèse (2), et comme c_0 est séparable, u est application faiblement compacte, donc (lemme 1) la suite des x'_i est bien relativement $\sigma(E', E'')$-compacte.

COROLLAIRE 2. *Tout espace quotient séparable de $C(K)$ (K, espace compact stonien) est réflexif. Un espace séparable de dimension infinie ne peut être facteur direct de $C(K)$.*

La première partie du corollaire résulte aussitôt du corollaire 1. La deuxième partie résulte de la première, compte tenu du fait qu'un espace réflexif de dimension infinie ne peut être facteur direct d'un espace $C(K)$ (corollaire de la proposition 5).

Le corollaire 2 précise de beaucoup le résultat bien connu que l'espace c_0 des suites de nombres complexes qui tendent vers zéro n'est pas facteur direct dans son bidual, espace de toutes les suites bornées. Plus généralement, si K est un espace compact métrisable infini, $C(K)$ n'est pas facteur direct dans son bidual, car $C(K)$ est séparable, et son bidual s'identifie à l'espace des fonctions continues sur un espace compact stonien (car le bidual s'identifie à un espace $L^\infty(\mu)$, comme dual de l'espace $\mathfrak{M}^1(K)$ qui s'identifie à un espace $L^1(\mu)$). Plus généralement encore, on a le

COROLLAIRE 3. *Soit K un espace compact quelconque, E un espace quotient de $C(K)$. Alors l'espace de Banach E'', bidual de E, satisfait aux conditions du*

lemme 8. En particulier, E'' ne peut être séparable que si E'', donc E, est réflexif. Et si E est séparable, E ne peut être facteur direct dans son bidual que si E est réflexif.

En effet, E'' est alors isomorphe à un espace quotient du bidual de $C(K)$, or ce dernier s'identifie à l'espace des fonctions continues sur un certain espace stonien, et satisfait par suite aux conditions du lemme 8.

Remarquons que le corollaire 3 nous donne d'autres propriétés spéciales aux espaces quotients d'espaces du type $C(K)$, s'ajoutant à celles obtenues aux §§ 2 et 3.

Du corollaire 3, on déduit aussi qu'un espace de Banach non réflexif qui est séparable, et isomorphe à un dual fort d'espace de Banach, ne peut être isomorphe à un quotient d'espace $C(K)$. On sait en effet qu'un dual E d'espace de Banach est facteur direct dans son bidual.

4.2 Sous-espace et espaces quotients de c_0. Soit c_0 l'espace des suites de nombres complexes qui tendent vers zéro, muni de la norme uniforme. Nous allons montrer que les sous-espaces et espaces quotients de c_0 ont, entre autres, toutes les propriétés envisagées aux §§ 1, 2, 3, pour les espaces $C(K)$.

Notons d'abord que le dual de c_0, qui s'identifie à l'espace l^1 des suites sommables, est séparable, donc il en est de même du dual d'un sous-espace ou d'un espace quotient de c_0. D'autres part, on a la facile

PROPOSITION 15. *Soit E un espace localement convexe complet dont les parties bornées sont faiblement métrisables* (cela signifie, si E est un espace de Banach, que son dual fort est séparable). *Alors:*

(1) *E jouit de la propriété D* (définition 4);

(2) *E jouit de la propriété* R. D.-P. (définition 3) *et même de la propriété plus forte: toute application linéaire u de E dans un espace localement convexe séparé complet qui transforme suites faiblement convergentes en suites fortement convergentes, transforme parties bornées en parties fortement relativement compactes.*

(3) *E peut ne pas jouir de la propriété* D.-P. (définition 1). *Mais s'il satisfait à cette propriété, alors toute application linéaire continue de E dans un espace localement convexe F qui transforme parties bornées de E en parties relativement faiblement compactes de F, transforme même parties bornées de E en parties fortement relativement compactes de F; et toute partie équicontinue de E' qui est relativement compacte pour $\sigma(E', E'')$ est relativement compacte pour la topologie forte.*

Démonstration. (1) a déjà été remarqué au §3, exemple (c).

(2) Il suffit de montrer que sous les conditions de l'énoncé (2), la restriction de u à une partie bornée convexe cerclée A de E est continue pour la topologie faible sur A et la topologie forte sur F. Car u sera alors même uniformément continue de A faible dans F fort (cela résulte par exemple du résultat intermédiaire—en italiques—de la démonstration du lemme 3), et comme A est faiblement précompact, son image sera bien fortement précompacte. — Comme A faible est métrisable, il suffit de vérifier que u est continue sur A faible pour les suites,

or par hypothèse u transforme suites faiblement convergentes en suites fortement convergentes.

(3) Un espace réflexif séparable de dimension infinie satisfait aux conditions générales de la proposition 15, sans jouir de la propriété D.-P. Mais supposons que E jouisse de la propriété D.-P., montrons qu'alors toute application linéaire continue u de E dans un espace localement convexe F qui transforme parties bornées en parties relativement faiblement compactes, les transforme même en parties fortement relativement compactes. Cela résulte en effet de (2), car l'hypothèse D.-P. implique que u transforme suites faiblement convergentes en suites fortement convergentes. Enfin, il en résulte aussi qu'une partie équicontinue A' de E' qui est relativement $\sigma(E', E'')$-compacte, est relativement fortement compacte. C'est en effet un cas particulier de la proposition 1, où $\mathfrak{S}$ désigne l'ensemble des parties bornées de E (en tenant compte de la remarque (b) qui suit cette proposition).

Théorème 10. *Soit E une espace de Banach que est isomorphe à un sous-espace ou à un espace quotient de c_0. Alors E jouit des propriétés* D.-P., R.D.-P. *et D, toute partie de E' qui est relativement $\sigma(E', E'')$-compacte est relativement fortement compacte, et toute suite dans E' qui est suite de Cauchy pour $\sigma(E', E'')$ est fortement convergente.*

De toutes façons, les propriétés D et R. D.-P. résultent de la proposition 15. La propriété D.-P. est facile à vérifier quand E est un espace quotient de c_0, car son dual sera isomorphe à un sous-espace vectoriel de l^1, donc toute partie de E' qui est $\sigma(E', E'')$-compacte est fortement compacte, puisqu'on sait que toute partie de l^1 qui est faiblement compacte (pour la topologie faible définie par le dual l^∞ de l^1) est fortement compacte. Le fait qu'une suite de Cauchy pour $\sigma(E', E'')$ converge fortement se voit alors de la même façon. Reste, dans le cas où E est un sous-espace de c_0, à montrer que E jouit de la propriété D.-P., et que toute suite de Cauchy pour $\sigma(E', E'')$ converge fortement. Pour la première propriété, il suffit de démontrer directement que pour toute suite (x_i) dans E qui tend vers zéro faiblement, et toute suite (x'_i) dans E' qui tend vers zéro pour $\sigma(E', E'')$, on a

$$\lim_i \langle x_i, x'_i \rangle = 0$$

(voir fin du No. 2 du §1). Et de même, pour la seconde propriété, il suffit de montrer que pour toute suite (x_i) dans E faiblement convergente vers zéro, et toute suite de Cauchy (x'_i) pour $\sigma(E', E'')$, on a $\lim \langle x_i, x'_j \rangle = 0$ uniformément en j (car alors l'application $x \to u(x) = (\langle x, x'_j \rangle)$ de E dans l'espace c des suites de nombres complexes qui tendent vers une limite transforme suites faiblement convergentes en suites fortement convergentes, donc est compacte (proposition 15) donc (lemme 2) l'image de la boule unité du dual de c par la transposée u' est compacte). Cela équivaut aussi manifestement à dire que pour toute suite de Cauchy (x'_i) pour $\sigma(E', E'')$, et toute suite (x_i) dans E faiblement convergente

vers zéro, on a $\lim \langle x_i, x'_i \rangle = 0$, condition qui inclut aussi la première des deux propriétés que nous avions en vue.

Tout revient donc à montrer qu'on ne peut avoir $|\langle x_i, x'_i \rangle| > \epsilon > 0$ pour tout i, quand (x_i) est une suite faiblement convergente vers zéro dans E, et (x'_i) une suite de Cauchy pour $\sigma(E', E'')$ dans E'. Mais cela résulte aisément du fait que cette propriété est vraie quand E est identique à c_0, et du

Lemme 9. *Soit (x_i) une suite dans c_0 qui converge faiblement vers zéro sans converger fortement. Alors il existe une suite (y_i) extraite, telle que le sous-espace vectoriel fermé de c_0 engendré par les y_i soit isomorphe à c_0.*

Ce lemme précise un résultat de Banach [**1**, p. 194], et se démontre de la même façon. On peut se ramener immédiatement au cas où $\|x_i\| = 1$ pour tout i (en extrayant sinon une suite partielle (y_i) telle que

$$\lim_i \|y_i\|$$

existe et soit non nulle, et en remplaçant alors les y_i par les $z_i = y_i/\|y_i\|$). On peut alors par récurrence construire une suite (y_i) extraite de (x_i), et une suite strictement croissante (N_i) d'entiers naturels, telles qu'on ait

$$|y_i(j)| < (\tfrac{1}{4})^i \text{ pour } j < N_{i-1} \text{ ou } j > N_i \quad (N_0 = 0). \tag{1}$$

La possibilité de la récurrence est évidente: on prend d'abord y_1 arbitrairement, puis N_1 tel que $j > N_1$ implique $|y_1(j)| < \frac{1}{4}$. Supposant alors la construction faite jusqu'au rang n (les relations (1) et (2) étant supposées satisfaites pour les indices i, j, i' qui sont $< n$), on continue en choisissant y_{n+1} tel que l'on ait $|y_{n+1}(j)| < (\frac{1}{4})^{n+1}$ pour $j < N_n$, puis N_{n+1} tel que l'on ait $|y_{n+1}(j)| < (\frac{1}{4})^{n+1}$ pour $j < N_{n+1}$.

Soit ϕ le sous-espace de c_0 formé des éléments dont un nombre fini seulement de cordonnées sont non nulles, muni de la norme induite par c_0, et pour $\lambda = (\lambda_i) \in \phi$, posons

$$u(\lambda) = \sum \lambda_i y_i.$$

On a ainsi une application linéaire de ϕ dans c_0, il suffit de montrer que c'est un isomorphisme dans, car alors l'isomorphisme $\tilde{u}$ de c_0 dans c_0 qui prolonge u aura évidemment pour image le sous-espace vectoriel fermé de c_0 engendré par les y_i. Or je dis qu'on a en effet, pour $\lambda \in \phi$:

$$\tfrac{2}{3}\|\lambda\| < \|u(\lambda)\| < \tfrac{4}{3}\|\lambda\|.$$

L'inégalité de droite équivaut à: $|u(\lambda).(j)| < \frac{4}{3}$ pout tout j, quand $\|\lambda\| < 1$. Or pour j donné, il existe un indice i tel que $N_{i-1} < j < N_i$, on aura alors

$$u(\lambda) . (j) = \sum_{i' (i' \neq i)} \lambda_{i'} y_{i'}(j) + \lambda_i y_i(j),$$

d'où, en vertu de (1), et puisque $\|\lambda\| < 1$, $\|y_i\| = 1$, $\|y_{i'}\| = 1$:

$$|u(\lambda) . (j)| < \sum_{i'} (\tfrac{1}{4})^{i'} + 1 = \tfrac{4}{3}.$$

Pour démontrer l'inégalité de gauche, on note que pour λ donné, existe un indice α tel que $|\lambda_\alpha| = ||\lambda||$, puis un indice j tel que $|y_\alpha(j)| = ||y_\alpha|| = 1$. En vertu de (1), on a forcément $N_{\alpha-1} < j < N_\alpha$. On a alors

$$||u(\lambda)|| \geqslant |u(\lambda) \,.\, (j)| = \Big| \sum_{i \neq \alpha} \lambda_i y_i(j) + \lambda_\alpha y_\alpha(j) \Big| \geqslant |\lambda_\alpha y_\alpha(j)| - \Big| \sum_{i \neq \alpha} \lambda_i y_i(j) \Big|$$

d'où, compte tenu de (1) et de $|\lambda_\alpha| = ||\lambda||$, $|y_\alpha(j)| = 1$:

$$||u(\lambda)|| \geqslant ||\lambda|| - ||\lambda|| \sum (\tfrac{1}{4})^i = \tfrac{2}{3}||\lambda||, \qquad \text{cqfd.}$$

Corollaire du théorème 10. *Soit E un espace de Banach isomorphe à un sous-espace vectoriel ou un espace quotient de c_0. Toute application linéaire faiblement compacte de E dans un espace localement convexe F est compacte. Si F est un espace quotient d'un espace $C(K)$, ou plus généralement, si F est un espace de Banach qui jouit de la propriété D, ou* R. D.-P., *alors tout forme bilinéaire continue sur $E \times F$ est compacte* (définition 5).

La première partie du corollaire résulte de la proposition 15, la deuxième s'obtient en identifiant la forme bilinéaire à une application linéaire de F dans le dual fort G de E, et en utilisant les propriétés de G énoncées dans le théorème 10.

Bibliographie

1. S. Banach, *Théorie des opérations linéaire* (Varsovie, 1932).
2. N. Bourbaki, *Algèbre* (Actualitiés sci. et industr., Paris, 1948), Chap. 3.
3. J. Dieudonné et L. Schwartz, *La dualité dans les espaces* $(\mathfrak{F})$ *et* $(\mathfrak{LF})$, Annales de Grenoble, *1* (1949), 61–101.
4. J. Dieudonné, *Sur les espaces de Köthe*, Journal d'Analyse Mathématique, *1* (Jérusalem, 1951), 81–115.
5. ———, *Sur la convergence des suites de mesures de Radon*, Anais da Acad. Bras. Ciencias, *23* (1951), 21–38.
6. N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc., *44* (1938), 305–356; in particular, Theorems 60, 75.
7. N. Dunford et J. Pettis, *Linear transformations on summable fonctions*, Trans. Amer. Math. Soc., *47* (1940), 323–392.
8. A. Grothendieck, *Critères dénombrables de compacité dans les espaces fonctionnels généraux*, Amer. J. Math., *74* (1952), 168–186.
9. ———, *Critères généraux de compacité dans les espaces localement convexes généraux*, C. R. Acad. Sci., Paris, *231* (1950), p. 217–240.
10. ———, *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques*, C. R. Acad. Sci., Paris, *233* (1951), 1556–1558.
11. S. Kakutani, *Concrete representation of abstract* (L)*-spaces*, Ann. Math., *42* (1941), 523–537.
12. L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc., *68* (1950), 28–46.
13. R. Philipps, *On linear transformations*, Trans. Amer. Math. Soc., *48* (1940), 525.
14. M. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., *41* (1937), 375–481.

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SOME MULTIPLICATIVE FUNCTIONALS

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This note concerns itself primarily with the representation of continuous multiplicative functionals on L_2 types of rings or Banach algebras to the real or complex fields where convolution is taken as the ring multiplication. In a recent publication [1] such functionals were studied for the continuous function ring $C(S)$ over a compact space S. It was shown that for each such multiplicative functional M there is an associated countable compactum, $D(M)$, termed a determining set in S, such that the values of $x(s)$ on $D(M)$ alone, fix $M(x)$ in the real case and $M|x|$ in the complex case. For the case considered in the present work, a similar result is valid except that a finite set enters in the role of $D(M)$.

For a Banach algebra the maximal ideals are associated with continuous functionals common to the family of linear functionals and to the family of multiplicative functionals. The first family bears on the underlying Banach space and has of course been extensively investigated. The results below and earlier results [1], study the second, hitherto neglected, family, which is associated with the underlying multiplicative semi-group. It seems promising also to consider our results from the viewpoint of a linear representation theory of this multiplicative semi-group. In this sense our work yields the representations of degree 1.

Suppose G is a compact Abelian group. We write G' for its discrete character group and use R and K for the real and complex fields respectively. We employ F to stand for either R or K. Let L_2 (G, F) be the ring of functions $x \sim x(g \mid G)$ with multiplication designated by a star and defined by convolution, i.e.,

$$(x * y)(h) = \int_G x(g) y(hg^{-1})\, dg. \tag{1}$$

Let $C^\circ(S, F)$, S discrete, be the Banach algebra of functions on S to F, vanishing except on a denumerable subset at most, and such that the function values converge to 0. The norm is that induced by $C(S, F)$ and the ring multiplication is pointwise multiplication of functions. $L_p(S, F)$ is the obvious ring with pointwise multiplication and elements designated by capitals, that is, $X \sim X(s \mid S)$. Plainly the elements of $L_p(S, F)$ are in $C^\circ(S, F)$. We use *countable* to cover either *finite* or *denumerable*. For convenience we quote two results of [1] that intervene in the sequel. We assume $M|x|$ is not identically 0 or 1.

THEOREM A. *If M is a norm continuous multiplicative functional on $C(Q, R)$ to R, where Q is Hausdorff compact, then $M(x)$ is determined by the values of $x(q)$ on a countable compactum D and $|M(x)|$ has the representation*

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$$|M(x)| = M|x| = \prod_{d \in D} |x(d)|^{\mu(d)},$$

where $\mu(d) > 0$.

THEOREM B. *If M is a norm continuous multiplicative functional on $C(Q, K)$ to K, where Q is Hausdorff compact, then $M|x|$ is determined by the values of $|x(q)|$ on a countable compactum D and $M|x|$ has the representation.*

$$M|x| = \exp \sum_{d \in D} \{\mu(d) + i\,\nu(d)\} \log |x(d)|,$$

where $\mu(d) > 0$ and $\nu(d)$ is unrestricted.

Since Theorem B is stated without proof in [1, p. 569] we sketch the demonstration. First $M|x| = |M|x|| \exp iF|x|$. Then $|M|x||$ is the transformation denoted by M_0 in [1] and has the determining set D. Evidently $F|x|$ is distributive. The association of a regular measure ν and the arguments concerning the measure adduced in [1, Theorem 3] apply here except of course that the inferences from $F|x_n| \to \infty$ do not differ from those from $F|x_n| \to -\infty$. The conclusion then is that ν may be of either sign or 0 and is concentrated on the countable compactum, D_i. The proof that $D \supset D_i$ is immediate, for otherwise $\{x^n\}$ exists with $x^n \to x$, where $x(s)$ vanishes at some point of D_i but is bounded away from 0 on D. Then $M|x_n|/|M|x_n||$ converges while $\exp iF|x_n|$ does not, a manifest absurdity.

We now give the two theorems fundamental for our conclusions.

THEOREM 1. *If M is a norm continuous single-valued multiplicative functional on E to K where E is either $C^{\circ}(S, K)$ or $L_p(S, K)$, $p > 1$, S discrete, then there is a finite set D in S and sets of complex numbers $\{\mu(d) + i\nu(d) | \mu(d) > 0, d \in D\}$ and integers $\{n(d) | D\}$ such that*

$$M(X) = \exp \left(\sum_{d \in D} (\mu(d) + i\,\nu(d)) \log |X(d)| + i\,n(d) \arg X(d)\right). \tag{2}$$

THEOREM 2. *If M is a norm continuous multiplicative functional on E to R where E is either $C^{\circ}(S, R)$ or $L_p(S, R)$ $p > 1$, S discrete, then there are a finite set D in S and real numbers*

$$\{\mu(d) \mid \mu(d) > 0, d \in D\}, \quad \{n(d) \mid n(d) = 0, \text{or } n(d) = 1\}$$

such that

$$M(X) = \prod_{d \in D} (|X(d)|^{\mu(d)} (\operatorname{sgn} X(d))^{n(d)}). \tag{3}$$

We consider first the proof of Theorem 1.

We tacitly assume below that the trivial cases $M(X) = 0$ or $M(X) = 1$ for all X are excluded. Let σ designate a finite subset of S. Write $1(\sigma)$ for the element of E which is 1 on σ and vanishes on the complement of σ. For arbitrary $X \in E$ let $Q(X) = \{s \mid X(s) \neq 0\}$. Suppose $Q(X)$ is denumerable. Order the finite subsets $\sigma(X)$ of $Q(X)$ by inclusion so $\{\sigma \mid \sigma = \sigma(X)\}$, for fixed X, is a directed set. Plainly $X\,1(\sigma)$ converges in the norm to X. Hence, since $M(X) \neq 0$ for say

$X = X_0$, there is a σ' in the collection $\{\sigma \mid \sigma = \sigma(X_0)\}$ for which $M(X_0 1(\sigma')) \neq 0$. If $Q(X)$ is finite, σ' may be taken as $Q(X)$ itself. Since $1(\sigma) = (1(\sigma))^2$ it follows that

$$M(X_0 1(\sigma')) = M(X_0 1(\sigma'))M(1(\sigma')).$$

Accordingly $M(1(\sigma')) = 1$. Therefore for arbitrary $X \in E$ we have

$$M(X) = M(1(\sigma'))M(X) = M(X 1(\sigma')). \tag{4}$$

Thus we need consider the values of M on σ' alone; so we are reduced to consideration of multiplicative functionals over $C(\sigma', K)$.

Suppose σ' consists of N points. Then

$$C(\sigma', K) = \prod_{j=1}^{N} K_j. \tag{5}$$

If $Z = (Z_1, \ldots, Z_N) \in C(\sigma', K)$ we refer to Z_j as the jth coordinate of Z. Write Z^j as the element of $C(\sigma', K)$ whose jth coordinate is Z_j and whose other coordinates are 1. We have

$$M(Z) = \prod_{j=1}^{N} M(Z^j). \tag{6}$$

Evidently for fixed j, $M(Z^j)$ *may be considered as on* K *to* K. Accordingly, suppose $N = 1$ and let $W = \rho e^{i\theta}$. We have then

$$M(W) = M(\rho)M(e^{i\theta}). \tag{7}$$

It is easy to see from the continuity condition that $M(\rho) = \exp(\mu + i\nu)\log \rho$, where $\mu > 0$ but ν is an unrestricted real number. For positive integers k and N and $\theta = 2\pi k/N$, $(M(e^{i\theta}))^N = M(1) = 1$. Hence $|M(e^{2k\pi i/N})| = 1$. Appeal to continuity establishes $|M(e^{i\theta})| = 1$ for arbitrary θ. The single-valuedness and continuity requirements on M imply now that M is a homomorphism on the topological group of the circle, P, into itself; that is to say that M is a character of P. It is well known then that $M(e^{i\theta}) = e^{in\theta}$ for n integral. In view of (6) the representation (2) for $N > 1$ is now fully verified.

The demonstration of Theorem 2 proceeds along similar lines. The significant part of the proof is the analogue of (5) with $C(\sigma', R)$ replacing $C(\sigma', K)$. Then a direct argument (or appeal to Theorem A, since σ' is compact) yields (3).

In the interests of completeness we note the effect of changing the continuity requirement on M.

THEOREM 3. *If M is a weakly continuous multiplicative functional on $C(Q, R)$ to R, where Q is Hausdorff compact, then D is finite and $M(x)$ has the representation* (3). *For R replaced by K the set D for $M|x|$ is finite and the representation for $M|x|$ falls under* (2).

Every functional continuous in the weak topology is surely continuous in the norm topology. Accordingly, the M's consistent with our hypotheses form a subset of those described in Theorem A and in Theorem B. Suppose that the

determining set D could be infinite. Let $e(s) = 1$ for all s in S. Plainly $M(e) = 1$. Any weak neighbourhood of e is of the form

$$N(e, \sigma, \epsilon) = \{x \mid |x(s) - 1| < \epsilon, s \in \sigma\}$$

where σ is a finite subset of S. Evidently D contains a point, d_σ, not in σ. Let x_σ be a continuous function vanishing at d_σ, taking on the value 1 on σ and otherwise subject to $0 \leq x_\sigma(s) \leq 1$. Then $x_\sigma \in N(e, \sigma, \epsilon)$. Yet $M(x_\sigma) = 0$. Since σ and ϵ are arbitrary this shows M cannot be continuous in the weak topology if D is nonfinite. If D consists of a single point M satisfies the conditions of the theorem whence by combination D can be taken as a finite point set.

Let $I = \{t \mid 0 \leq t \leq 1\}$ in the sequel but interpret the elements of $L_2(I)$ as even periodic functions over $2I$ $(-1 \leq t \leq 1)$ *with convolutions over* $2I$. The functions $\{\psi_n(t) \mid \psi_0(t) = 2^{-1},\ \psi_n(t) = \cos n\pi t,\ n > 0\}$ constitute a complete orthogonal set for $L_2(I)$ and the expansion of x *in terms of* $\{\psi_n(t)\}$ we call the Fourier cosine series expansion.

THEOREM 4. *If M is a norm continuous multiplicative functional on the real Banach algebra $L_2(I)$ to R where ring multiplication in $L_2(I)$ is interpreted as convolution, then, for some finite set of integers, D,*

$$M(x) = \prod_{d \in D} |X(d)|^{\mu(d)} (\operatorname{sgn} X(d))^{n(d)}, \tag{8}$$

where $\mu(d) > 0$, $n(d) = 0$ or 1 and $\{X(n) \mid n = 0, 1, 2, \ldots\}$ are the coefficients in the Fourier cosine series expansion of x.

Let

$$x(t) \sim \sum_{j=0}^{\infty} X(j)\psi_j(t) \tag{9}$$

Then of course $\{X(j)\} \in l_2$. In view of the Parseval identity,

$$\begin{aligned} (x*y)(t) &= \int_{-1}^{1} x(\tau)y(t-\tau)\,d\tau \\ &\sim \sum_{j=0}^{\infty} (X(j)Y(j))\psi_j(t). \end{aligned} \tag{10}$$

The correspondence $x \leftrightarrow \{X(j)\}$ is a linear homeomorphism of $L_2(I)$ onto l_2. Indeed, it is compounded of $x \leftrightarrow 2^{\frac{1}{2}} x \leftrightarrow \{2^{-\frac{1}{2}} X(0),\ X(j \mid j \geq 1)\} \leftrightarrow \{X(j \mid j \geq 0)\}$, where the first map merely recognizes that the norm is taken over I and not $2I$, while the next map is a linear isometry etc. Accordingly, Theorem 2 can be applied in combination with (10) to establish (8).

It is well known [2] that a Fourier transform T can be defined on $L_2(G, K)$ to $L_2(G', K)$. Indeed $X(g') = (Tx)(g')$ is simply the coefficient in the development of x in terms of the character g' and so corresponds exactly to $X(j)$ in (9). The inverse Fourier transform T' from $L_2(G', K)$ to $L_2(G, K)$ satisfies $T'Tx = x$. Furthermore it is known that T and T' are unitary and

$$T(x*y) = Tx \,.\, Ty = X\ Y. \tag{11}$$

THEOREM 5. *If M is a norm continuous single-valued multiplicative functional on the ring $L_2(G, K)$ to K, where ring multiplication is taken as convolution, then there is a finite set D in G', complex numbers $\{\mu(d) + i\nu(d) \mid \mu(d) > 0, d \in D\}$ and integers $\{n(d) \mid D\}$ such that*

$$M(x) = \prod_{d \in D} |(T(x))(d)|^{\mu(d)+i\nu(d)} \exp i \sum_{d \in D} n(d) \arg ((T(x))(d)). \tag{12}$$

Let

$$N(X) = M(T'X) = M(x).$$

We remark

$$N(XY) = M(T'(XY) = M(x*y) = M(x)M(y) = N(X)N(Y).$$

Thus N is multiplicative and single-valued on $L_2(G', K)$ to K. Moreover since x and X are related by a unitary transformation the norm continuity of M implies norm continuity of N and conversely. Accordingly, Theorem 1 may be invoked to yield the representation (12).

REFERENCES

1. D. G. Bourgin, *Multiplicative transformations*, Proc. Nat. Acad. Sci., *36* (1950), 564–570.
2. A. Weil, *L'integration dans les groupes topologiques et ses applications*, Actualités sci. et industr., *869* (Publications de l'Inst. de Math. de Clermont-Ferrand, IV), 1–158.

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INTEGRAL FUNCTIONS WITH NEGATIVE ZEROS

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1. Introduction. If $f(z)$ is an integral function of non-integral order with only real negative zeros, there is a close connection between the rates of growth of the function and of $n(r)$, the number of zeros of absolute value not exceeding r. The best known theorem is that of Valiron [**12**], which may be stated as follows.

THEOREM 1. *If $f(z)$ is an integral function with real negative zeros, of order less than 1, with $f(0) = 1$, the conditions*

$$\log f(r) \sim A\pi \csc \pi\rho\, r^{\rho}, \qquad r \to \infty,\ A > 0, \tag{1.1}$$

and

$$n(r) \sim Ar^{\rho} \tag{1.2}$$

are equivalent.

Either (1.1) or (1.2) implies that $f(z)$ is of order ρ, $0 < \rho < 1$, and from either condition it can be deduced [**1**; **5**] that

$$\log f(re^{i\theta}) \sim \pi A \csc \pi\rho\, e^{i\rho\theta} r^{\rho} \tag{1.3}$$

for $|\theta| < \pi$, uniformly in $|\theta| \leq \pi - \delta < \pi$.

When $\rho = \frac{1}{2}$, Theorem 1 implies, after a change of variable, a statement about a canonical product of order 1 with real zeros (not necessarily even).

THEOREM 2. *If $f(z)$ is a canonical product of order 1 with real zeros, the conditions*

$$\log |f(iy)| \sim \pi A |y|, \qquad |y| \to \infty, \tag{1.4}$$

and

$$n(r) \sim 2Ar \tag{1.5}$$

are equivalent.

There is another condition which was shown by Paley and Wiener [**8**, p. 70] to be equivalent to those of Theorem 2.

THEOREM 3. *Under the hypotheses of Theorem 2, if $f(0) = 1$, the condition*

$$\lim_{R\to\infty} \int_{-R}^{R} x^{-2} \log |f(x)|\, dx = -\pi^2 A \tag{1.6}$$

is equivalent to (1.4) *and* (1.5).

In terms of functions of order $\frac{1}{2}$, Theorem 3 becomes the following:

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THEOREM 4. *If $f(z)$ is of order $\frac{1}{2}$, all its zeros are real and negative, and $f(0) = 1$, the conditions*

$$\lim_{R\to\infty}\int_0^R x^{-3/2}\log|f(-x)|\,dx = -\pi^2 A, \tag{1.7}$$

$$\log f(r) \sim A\pi r^{\frac{1}{2}},$$

$$n(r) \sim A r^{\frac{1}{2}}$$

are equivalent.

My object is to investigate what becomes of Theorem 4 for a general order ρ, $0 < \rho < 1$. The result is as follows.

THEOREM 5. *If $f(z)$ is of order less than 1, all its zeros are real and negative, and $f(0) = 1$, the conditions* (1.1) *and (for any σ, $0 < \sigma < 1$)*

$$\int_0^r x^{-1-\sigma}\{\log|f(-x)| - \pi\cot\pi\sigma\, n(x)\}\,dx \sim \pi A(\rho-\sigma)^{-1}(\cot\pi\rho - \cot\pi\sigma)r^{\rho-\sigma} \tag{1.8}$$

are equivalent.

When $\sigma = \rho$, (1.8) is to be interpreted as (1.9), below. The conclusion implies in particular that $f(z)$ is of order ρ. For $\rho = \sigma = \frac{1}{2}$, Theorem 5 reduces to Theorem 4.

It is also true (and can be proved somewhat more simply) that the integral on the left-hand side of (1.8) is $O(r^{\rho-\sigma})$ if and only if $\log f(r) = O(r^\rho)$.

Special cases of (1.8) which are natural generalizations of (1.7) are

$$\int_0^\infty x^{-1-\rho}\{\log|f(-x)| - \pi\cot\pi\rho\, n(x)\}dx = -\pi^2 A\csc^2\pi\rho \qquad (\sigma = \rho),$$

$$\int_0^r x^{-3/2}\log|f(-x)|\,dx \sim \pi A(\rho-\tfrac{1}{2})^{-1}\cot\pi\rho\, r^{\rho-\frac{1}{2}} \qquad (\sigma = \tfrac{1}{2} < \rho), \tag{1.9}$$

$$\int_r^\infty x^{-3/2}\log|f(-x)|\,dx \sim \pi A(\tfrac{1}{2}-\rho)^{-1}\cot\pi\rho\, r^{\rho-\frac{1}{2}} \qquad (\sigma = \tfrac{1}{2} > \rho).$$

For $\rho \neq \frac{1}{2}$, we see from (1.9) that

$$\int_0^\infty x^{-1-\rho}\log|f(-x)|\,dx$$

converges if and only if

$$\int_0^\infty x^{-1-\rho} n(x)\,dx$$

converges, which is equivalent to $\sum r_n^{-\rho} < \infty$, where $-r_n$ are the zeros of $f(z)$. In this case, of course, $A = 0$.

A consequence of Theorem 5 is that (1.1) implies

$$\int_0^r x^{-1-\rho}\log|f(-x)|\,dx \sim \pi A\cot\pi\rho\log r \qquad (\rho \neq \tfrac{1}{2}),$$

$$\int_0^r x^{-1-\sigma} \log |f(-x)|\, dx \sim \pi A(\rho - \sigma)^{-1} \cot \pi\rho\, r^{\rho-\sigma} \qquad (\rho > \sigma),$$

$$\int_r^\infty x^{-1-\sigma} \log |f(-x)|\, dx \sim \pi A(\sigma - \rho)^{-1} \cot \pi\rho\, r^{\rho-\sigma} \qquad (\rho < \sigma).$$

We may compare these relations with Titchmarsh's result [**10**] that

$$\log |f(-x)| \sim \pi A \cot \pi\rho\, x^\rho$$

in a set of unit linear density; a converse theorem was given by Titchmarsh [**10**] and by Bowen and Macintyre [**2**].

Theorem 3 was proved by Paley and Wiener by using Wiener's general Tauberian theorems; a proof that (1.6) implies (1.4), using methods from the theory of functions, was given by Levinson [**6**, p. 33], but no such proof of the converse appears to have been given previously. The proof of Theorem 5 incidentally contains a new proof of Theorem 3 by function-theory methods.

In Theorem 1 the inference (1.2) implies (1.1) is easy; the converse is more difficult. It was first proved by Valiron [**11**], and later by Titchmarsh [**10**] and by Paley and Wiener [**8**], by Tauberian methods; proofs depending more on the theory of functions have been given by Valiron [**12**], Pfluger [**9**], Levinson [**6**] for $\rho = \frac{1}{2}$, Delange [**4**; **4a**], Bowen [**1**], and Heins [**5**]; the last two are the simplest. For further developments along the lines of Theorem 1 see the papers cited and also Bowen and Macintyre [**2**; **3**] and Noble [**7**].

2. Theorem 5: first part. We begin by proving that (1.8) implies (1.1). Consider the integral

$$I = \int_C r(r - z)^{-1} z^{-1-\sigma} \log f(z)\, dz, \tag{2.1}$$

where C is the contour made up of the circle $|z| = R > r$, with a cut along the negative real axis from $z = -R$ to $z = 0$ and back again; the multiple-valued functions are to be positive for large positive values of z. Initially C has indentations to avoid the zeros of $f(z)$ and the origin, but the contributions of the indentations tend to zero with the diameters of the indentations, and we may disregard them. We also suppose that $-R$ is not one of the zeros of $f(z)$. The integrand is regular except for a pole at $z = r$, and consequently we have

$$I = -2\pi i r^{-\sigma} \log f(r). \tag{2.2}$$

To evaluate the integral along the cut we note that if we take $\arg f(z)$ to be zero for $x > 0$, we have $\arg f(-x) = \pi n(x)$, $x > 0$, on the upper side of the cut, and $\arg f(-x) = -\pi n(x)$ on the lower side. Hence the contribution of the cut is

$$2i \int_0^R r(r + x)^{-1} \phi(x)\, dx,$$

where

$$\phi(x) = x^{-1-\sigma}\{\sin \pi\sigma \log |f(-x)| - \pi \cos \pi\sigma\, n(x)\}.$$

The integral around the circle approaches zero, at least as $R \to \infty$ through an appropriate sequence of values, because if $f(z)$ is of order λ, say, for any positive ϵ we have $\log |f(z)| < R^{\lambda+\epsilon}$ for all large R, $\log |f(z)| > - R^{\lambda+\epsilon}$ for a sequence of values of R tending to ∞; and $|\arg f(z)| < R^{\lambda+\epsilon}$ because $n(t) = O(t^{\lambda+\epsilon})$ and so

$$\arg f(z) = \Im \log f(z) = y \int_0^\infty \frac{n(t)\, dt}{(t+x)^2 + y^2} = O(R^{\lambda+\epsilon})$$

(cf. Valiron [**12**], Bowen and Macintyre [**2**]). Hence

$$\int_0^\infty r(r+x)^{-1}\phi(x)\, dx = -\pi r^{-\sigma} \log f(r), \tag{2.3}$$

where the integral is to be understood as

$$\lim \int_0^R$$

when $R \to \infty$ through a certain sequence of values.

If $\rho = \sigma$,

$$\int_0^\infty \phi(x)\, dx$$

converges and (since $r/(r+x)$ is monotonic) we may let $r \to \infty$ under the integral sign in (2.3) to obtain (1.1) from (1.8).

If $\rho < \sigma$, put

$$\Phi(x) = \int_0^x \phi(t)\, dt;$$

then (1.8) gives us

$$\Phi(x) \sim Br^{\rho-\sigma}, \qquad B = \pi A(\rho - \sigma)^{-1}(\cot \pi\rho - \cot \pi\sigma).$$

By (2.3) we have

$$-\pi r^{-\sigma} \log f(r) = \int_0^\infty r(r+x)^{-1} d\Phi(x) = \int_0^\infty r(r+x)^{-2}\Phi(x)\, dx,$$

and since $\Phi(x) \sim Bx^{\rho-\sigma}$,

$$\int_0^\infty r(r+x)^{-2}\Phi(x)\, dx \sim B \int_0^\infty rx^{\rho-\sigma}(r+x)^{-2}\, dx = Br^{\rho-\sigma}\pi(\sigma - \rho)\csc \pi(\sigma - \rho),$$

and (1.1) follows. If $\rho > \sigma$ we write

$$\Phi(x) = \int_x^\infty \phi(t)\, dt$$

and proceed similarly.

3. Theorem 5: second part. We now show that (1.1) implies (1.8). By (1.3), (1.1) implies

$$\log f(z) \sim A\pi z^\rho \csc \pi\rho, \qquad -\pi < \theta < \pi, \tag{3.1}$$

uniformly in $|\theta| < \pi - \delta < \pi$. Consider the integral

$$-i \int_C z^{-1-\sigma} \log f(z)\, dz$$

over the contour used in §2. The integrand is regular inside the contour and so the integral is zero. The integral along the cut is

$$2\int_0^R x^{-1-\sigma}\{\sin \pi\sigma \log|f(-x)| - \pi \cos \pi\sigma\, n(x)\}\, dx.$$

The integral around the circle is

$$\int_{-\pi}^{\pi} z^{-\sigma} \log f(z)\, d\theta. \tag{3.2}$$

By (3.1), if we can let $R \to \infty$ under the integral sign in (3.2), we shall have

$$\lim_{R\to\infty} R^{\sigma-\rho} \int_{-\pi}^{\pi} z^{-\sigma} \log f(z)\, d\theta = 2A\pi(\rho - \sigma)^{-1}\csc \pi\rho \sin \pi(\rho - \sigma), \tag{3.3}$$

which will establish (1.8). Now the convergence in (3.1) is uniform in $(-\pi + \delta, \pi - \delta)$, and so

$$\lim_{R\to\infty} R^{\sigma-\rho} \int_{-\pi+\delta}^{\pi-\delta} z^{-\sigma} \log f(z)\, d\theta = 2\pi A(\rho - \sigma)^{-1} \csc \pi\rho \sin(\pi - \delta)(\rho - \sigma). \tag{3.4}$$

The remainder of the integral contributes

$$R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right)\{\log|f(Re^{i\theta})| \cos \sigma\theta + \arg f(Re^{i\theta}) \sin \sigma\theta\}\, d\theta. \tag{3.5}$$

The part involving $\arg f(Re^{i\theta})$ is $O(\delta)$ as $\delta \to 0$, uniformly in R, since $n(R) = O(R^\rho)$ implies $\arg f(Re^{i\theta}) = O(R^\rho)$ as before.

By Jensen's theorem and Theorem 1,

$$R^{-\rho} \int_{-\pi}^{\pi} \log|f(Re^{i\theta})|\, d\theta = 2\pi \int_0^R t^{-1}n(t)\, dt \to 2\pi A/\rho,$$

and by (3.1),

$$R^{-\rho} \int_{-\pi+\delta}^{\pi-\delta} \log|f(Re^{i\theta})|\, d\theta \to 2\pi A\rho^{-1} \sin(\pi - \delta)\rho \csc \pi\rho;$$

so

$$R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right) \log|f(Re^{i\theta})|\, d\theta \to 2\pi\rho^{-1}\{1 - \sin(\pi - \delta)\rho \csc \pi\rho\} = O(\delta). \tag{3.6}$$

Furthermore, the parts of (3.5) and (3.6) involving $\log^+|f(Re^{i\theta})|$ are uniformly $O(\delta)$ since $\log^+|f(Re^{i\theta})| = O(R^\rho)$ uniformly in θ. Then

$$\left|R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right) \log^-|f(Re^{i\theta})| \cos \sigma\theta\, d\theta\right| < \left|R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right) \log^-|f(Re^{i\theta})|\, d\theta\right| = O(\delta).$$

Thus the part of the left-hand side of (3.3) omitted from (3.4) is uniformly $O(\delta)$, and hence (3.3) is true.

References

1. N. A. Bowen, *A function-theory proof of Tauberian theorems on integral functions*, Quart. J. Math., *19* (1948), 90–100.
2. N. A. Bowen and A. J. Macintyre, *Some theorems on integral functions with negative zeros*, Trans. Amer. Math. Soc., *70* (1951), 114–126.
3. ———, *An oscillation theorem of Tauberian type*, Quart. J. Math. (2), *1* (1950), 243–247.
4. H. Delange, *Sur les suites de polynomes ou de fonctions entières à zéros réels*, Ann. sci. Ec. norm. sup. Paris. (3), *62* (1945), 115–183.

4a. ———, *Un théorème sur les fonctions entières à zéros réels et négatifs*, J. Math. pures appl. (9), *31* (1952), 55–78.

5. M. Heins, *Entire functions with bounded minimum modulus; subharmonic function analogues*, Ann. Math. (2), *49* (1948), 200–213.
6. N. Levinson, *Gap and density theorems* (New York, 1940).
7. M. E. Noble, *Extensions and applications of a Tauberian theorem due to Valiron*, Proc. Cambridge Phil. Soc., *47* (1951), 22–37.
8. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain* (New York, 1934).
9. A. Pfluger, *Die Wertverteilung und das Verhalten von Betrag und Argument einer speziellen Klasse analytischer Funktionen*, I, II, Comment. Math. Helv., *11*, (1938), 180–214; *12* (1939), 25–65.
10. E. C. Titchmarsh, *On integral functions with real negative zeros*, Proc. London Math. Soc. (2), *26* (1927), 185–200.
11. G. Valiron, *Sur les fonctions entières d'ordre fini et d'ordre nul, et en particulier les fonctions à correspondance régulière*, Ann. Fac. Sci. Univ. Toulouse (3), *5* (1914), 117–257.
12. ———, *Sur un théorème de M. Wiman*, Opuscula Mathematica A. Wiman Dedicata (Lund, 1930), 1–12.

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ON THE NUMBER OF SYMMETRY TYPES OF BOOLEAN FUNCTIONS OF n VARIABLES

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1. Introduction. In recent years Boolean Algebra has come to play a prominent role in the analysis and synthesis of switching circuits [**1**; **4**]. One general synthesis problem in which this algebra has proved useful is the following. Let there be given n input leads each of which can assume one of two possible states. It is desired to construct a network with these n input leads and a single output lead also capable of assuming either of two states. Furthermore, the state of the output lead for each of the 2^n states of the input leads is prescribed. Techniques are now available for solving this problem and under various assumptions as to the meaning of "best," techniques for finding the "best" network are also available [**1**].

The operation performed by the above network can be described by a Boolean function of n variables. Thus if the variables $x_1, x_2, \ldots, x_n$ represent the states of the n input leads (each x takes values 0 or 1), then the state of the output lead can be given by a Boolean function $f(x_1, x_2, \ldots, x_n)$. Specifying the function f determines the synthesis problem and under suitable restrictions leads to the synthesis of a definite physical network to realise f. From a physical point of view, however, it is immaterial how the n input leads are labelled or which of the two states any lead can assume is called zero or one. Therefore any Boolean function that can be obtained from f by permuting and (or) complementing one or more variables must be regarded as corresponding to the same physical network as f. It is convenient to define two Boolean functions of n variables to be of the same type if one of the functions can be obtained from the other by the process of permuting and (or) complementing one or more variables. There are then only as many distinct physical switching networks of the sort described above as there are types of Boolean functions of n variables. It is the purpose of this paper to enumerate the types of Boolean functions.

The argument to be used in determining N_n, the number of types of Boolean functions of n variables, is as follows. In §2 it is noted that there are only $\mu = 2^{2^n}$ possible Boolean functions of n variables and that each of these μ functions can be written as a linear combination of a certain set of 2^n simple Boolean functions, s_ν. The operations of permuting and (or) complementing one or more of the n variables of a Boolean function constitute a finite group, O_n, simply isomorphic with the hyper-octahedral group. Under the operations of O_n, the s_ν are permuted among themselves, as are also the μ Boolean functions of n variables. The permutations of the latter furnish a representation, D, of

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O_n, which is shown to be reducible containing the identity representation N_n times. The theory of group characters then yields

$$N_n = \frac{1}{2^n n!} \sum n_C \chi_C$$

where n_C is the number of elements of class C of O_n, χ_C in the character of class C in the representation D, and the summation is over all classes of O_n. Similar considerations give rise to a formula for $N_n^{(m)}$, the number of types of Boolean functions that are a linear sum of exactly m of the functions s_v. To make computations from the formulae of §2, it is necessary to know n_C, χ_C and quantities $\lambda_i^{(C)}$ which serve to define the cycle structure of the permutation of the s_v induced by any element of class C of O_n. In §3 these quantities are determined. A resumé of the computational procedure is given in §4 and results of computations performed are presented.

2. Formulae for N_n and $N_n^{(m)}$. It is well known that any Boolean function of n variables can be uniquely expanded in the form

$$f_u(x_1, x_2, \ldots, x_n) = \sum_{v=0}^{2^n-1} \epsilon_{uv} s_v \tag{1}$$

where the ϵ_{uv} can take values zero or one and the s_v are the 2^n simple Boolean functions

$$s_0 = x_1 x_2 \ldots x_n, \quad s_1 = x_1 x_2 \ldots x_n', \quad \ldots, \quad s_{2^n-1} = x_1' x_2' \ldots x_n',$$

i.e., the functions obtained by priming the product $x_1 x_2 \ldots x_n$ in all possible 2^n ways. (The prime is used to denote complementation.) Since each ϵ can assume one of two values, there are only $\mu = 2^{2^n}$ possible Boolean functions of n variables so that $u = 0, 1, \ldots, \mu - 1$.

Agreeing to arrange the x's of any s_v so that their subscripts are in natural order, we can represent any s by an n-position symbol consisting of zeros or ones, the ith position of the symbol being zero if x_i is not primed and one otherwise. We agree to label the s's so that the symbol for s_v is the integer v expressed in binary notation. Similarly each f_u can be specified by the 2^n zeros or ones, ϵ_{uv}, and we order the f's so that u is the number whose binary expression is $\epsilon_{u0}\, \epsilon_{u1} \ldots \epsilon_{u[2^n-1]}$.

It is readily seen that the operations of permuting and (or) complementing the variables of a Boolean function form a finite group, O_n, the multiplication law being defined by successive application of the operators to f. We adopt the customary cycle notation for permutations so that, for example, $\sigma = (123)(45)$ applied to f means replace x_1 by x_2, replace x_2 by x_3, etc. Complementation may be expressed by an operator N_i where i is written in binary notation. Thus N_{10011} applied to f means prime x_1, x_4, and x_5, and $N_i\sigma$ means first apply σ to f, then apply N_i.

We now define the complementation operator $N_{\sigma i}$ to mean the operator N_j where j is the binary expression obtained by applying the permutation σ to

the places of the binary symbol i. For example,

$$N_{(126)(54)101100} = N_{001011}\,,$$

the symbol in the first place being replaced by the symbol in the second place, etc. With this convention, the law $N_i\sigma = \sigma N_{\sigma i}$, $\sigma N_i = N_{\rho i}\sigma$, $\rho = \sigma^{-1}$ is readily established so that every element of O_n can be written in the form $N_i\sigma$. Since there are 2^n complementation operators N_i, and $n!$ permutation operators, σ, the order of O_n is $2^n n!$. The group is recognized as being simply isomorphic to the hyper-octahedral group [**5**; **6**], the group of symmetries of the hyper-octahedron in n-dimensional Euclidean space. This group is also the group of symmetries of the hyper-cube in n-space, and the permutations of the s_ν effected by the elements of O_n correspond to the permutations of the vertices of the hyper-cube under the various symmetry operations.

The totality of operations, H, of O_n which leave any particular f_u invariant form a subgroup of O_n of order h, say. H will possess $r = 2^n n!/h$ left cosets under O_n. It is easily shown then that operating on f_u by all the elements of O_n will result in exactly r distinct Boolean functions. These r functions are of one type and are all the functions of this type. The permutations of these r functions under the operations of O_n when written as permutation matrices furnish a representation of O_n of dimension r. This representation is just the permutation representation furnished by the left cosets of H and is therefore reducible containing the identity representation exactly once [**2**, p. 94].

Now the μ Boolean functions (1) are also permuted among themselves under the operations of O_n and these permutations when written as permutation matrices furnish a representation, D, of dimension μ of O_n. From the remarks of the preceding paragraph, it follows that D is reducible since it contains each of the r-dimensional representations once. D therefore contains the identity representation exactly N_n times, where N_n is the number of types of Boolean functions of n variables, and we can write

$$(2) \qquad N_n = \frac{1}{2^n n!} \sum n_C\, \chi_C\,.$$

Here n_C is the number of elements of class C of O_n, χ_C is the character of class C in the representation D, and the summation is over all classes of O_n.

Under the operations of O_n, the quantities s_ν are clearly permuted among themselves. It is easily shown, however, that two elements of the same class of O_n give rise to permutations of the s_ν that have the same cycle structure. We are thus led to investigate the number of f_u left invariant when the s_ν are permuted according to some fixed cycle structure, for this number is the character of the representation D of the class of O_n which permutes the s_ν according to this fixed cycle structure.

Let σ be a permutation of the s_ν into K cycles of length λ_i $(i = 1, 2, \ldots, K)$. We have

$$\sum_{1}^{K} \lambda_i = 2^n.$$

Consider now the matrix $\epsilon_{u\nu}$ of equation (1). The μ rows of this array are the binary representations of the integers from 0 to $\mu - 1$, and these rows may be labelled by the f_u. Similarly the columns may be labelled by the s_ν. On permuting the columns of the ϵ matrix according to σ, the rows considered as numbers expressed in binary form are no longer in natural order and their new order specifies the permutation of the f's induced by σ. Clearly only those f's will be left invariant which have either all zeros or all ones in the λ_i particular columns effected by the ith cycle of σ $(i = 1, 2, \ldots, K)$. Of the μ rows of ϵ, a fraction $2/2^{\lambda_i}$ have this property, so that there are

$$\mu \prod_{i=1}^{K} 2/2^{\lambda_i} = 2^K$$

f's left invariant under σ. We can therefore rewrite (2) in the form

$$N_n = \frac{1}{2^n n!} \sum 2^{K(C)} n_C \tag{3}$$

where $K(C)$ is the number of cycles in which the s_ν are permuted by any element of class C of O_n.

In a similar manner we can obtain a formula for the number of types of Boolean functions, $N_n^{(m)}$, that have exactly m non-zero terms in their expansion (1). Under the operations of O_n, these f's are permuted among themselves and these permutations written as matrices furnish a reducible representation of O_n. If the character of this representation is $\chi_C^{(m)}$, we have

$$N_n^{(m)} = \frac{1}{2^n n!} \sum n_C \, \chi_C^{(m)} . \tag{4}$$

To determine $\chi_C^{(m)}$ consider the rows of the matrix $\epsilon_{u\nu}$ of (1) corresponding to those

$$\binom{2^n}{m}$$

f's containing exactly m s's. Let σ be the permutation of the s_ν induced by an element of class C of O_n and let σ consist of cycles of length

$$\lambda_i^{(C)} \left(i = 1, 2, \ldots, K; \ \sum_{1}^{K(C)} \lambda_i^{(C)} = 2^n \right).$$

$\chi_C^{(m)}$ is the number of these rows left invariant on permuting the columns according to σ and is therefore the number of ways in which m can be obtained as a sum of terms taken from the series $\lambda_1, \lambda_2, \ldots, \lambda_K$, no term occurring more than once in any one sum. Thus $\chi_C^{(m)}$ is the coefficient of y^m in

$$\prod_{i=1}^{K(C)} (1 + y^a)$$

where $a = \lambda_i^{(C)}$. Equation (4) now becomes

$$N_n^{(m)} = \text{coefficient of } y^m \text{ in } \ \frac{1}{2^n n!} \sum n_C \prod_{i=1}^{K(C)} (1 + y^a) \tag{5}$$

where the sum is over all classes of O_n and the elements of class C effect a permutation of the s's with cycle structure $\lambda_i^{(C)}$.

Formula (5) has been given by Pólya [3] who has computed values of $N_n^{(m)}$ for $n = 1, 2, 3, 4$. Pólya, however, gives no means of determining the quantities n_C and $a = \lambda_i^{(C)}$. It is believed that formula (3) for N_n is new.

Equation (3) is a special case of the solution to the following more general enumeration problem. Each vertex of the hyper-cube in Euclidean n-space can be marked with one of p colors. Two such paintings of the hyper-cube are said to be of the same type if one can be obtained from the other by a symmetry operation of the hyper-cube. The number of types of paintings is

$$\frac{1}{2^n n!}\sum p^{K(C)} n_C.$$

3. Classes of O_n and the quantities n_C and $\lambda_i^{(C)}$. Details of the classes of O_n have been worked out by Young [6]. It will therefore suffice here to set down briefly a notation for the classes and a system for determining the class of a given element, $N_i\sigma$, of O_n.

Let $(ab\ldots)$ be a typical cycle of σ where $a, b, \ldots$ are certain of the symbols $1, 2, \ldots, n$. The complementation operator N_i will indicate that either an even or an odd number of the variables $x_a, x_b, \ldots$ are to be primed by the operation $N_i\sigma$. In the former case we refer to $(ab\ldots)$ as an e-cycle of the element $N_i\sigma$, in the latter case an o-cycle. With this terminology, the elements of O_n can be classified by the following scheme. Let σ consist of α_i cycles of length i so that

$$\sum_1^n i\alpha_i = n.$$

Let β_i be the number of the α_i cycles of length i that are e-cycles of $N_i\sigma$, so that the possible values of β_i are $\beta_i = 0, 1, 2, \ldots, \alpha_i$ $(i = 1, 2, \ldots, n)$. To every element of O_n there then corresponds a symbol

$$(\alpha_1, \alpha_2, \ldots \alpha_n;\ \beta_1, \beta_2, \ldots, \beta_n)$$

or $(\alpha; \beta)$ for short.

It is shown in [6] that two elements of O_n are in the same class if and only if they have the same $(\alpha; \beta)$ symbol. A simple calculation shows that the number of elements in the class $(\alpha; \beta)$ of O_n is

$$n_{(\alpha;\,\beta)} = n!\prod_{i=1}^{n}\frac{2^{(i-1)\alpha_i}}{\beta_i!\,(\alpha_i-\beta_i)!\,i^{\alpha_i}} \tag{6}$$

and the number of classes in O_n is

$$\sum\prod_{i=1}^{n}(\alpha_i+1)$$

where the sum is over all partitions of n.

We now inquire as to the cycle structure of the permutation of the s_v induced by an operation N_v of the class C of O_n. Since all elements of the class C permute

the s_ν in the same cycle structure, it will suffice to consider the effect of a particularly simple element of this class. We choose the element $N_i\sigma$ where the complementation operator N_i does not prime any of the variables permuted by the e-cycles of $N\sigma$ and where N_i primes only one variable from each set of variables permuted by the various separate o-cycles of $N\sigma$. The permutation of the s_ν induced by $N_i\sigma$ can best be studied by representing the s_ν by the numbers from 0 to $2^n - 1$ written in binary scale and listed in natural order in a column. The effect of $N_i\sigma$ on the s_ν is given by first permuting the columns of this array according to σ, and then in one column corresponding to each o-cycle interchanging the role of zero and one. The new array is again a list of the numbers from 0 to $2^n - 1$ in binary scale, and the new order of these numbers specifies the permutation of the s_ν effected by $N_i\sigma$. Suppose the cycles of σ are of length

$$\lambda_i \left(i = 1, 2, \ldots, K; \sum_1^K \lambda_i = n\right).$$

In the original array in any given row and in the λ_i columns corresponding to the ith cycle of σ, there will appear zeros and ones specifying in binary form a number, ξ_i, between 0 and $2^{\lambda_i} - 1$. We can accordingly specify the 2^n s_ν by K-place symbols

$$(\xi_1, \xi_2, \ldots, \xi_K), \qquad \xi_i = 0, 1, \ldots, 2^{\lambda_i} - 1.$$

The ith cycle of σ, whether an e-cycle or an o-cycle of $N_i\sigma$, has the effect of permuting the 2^{λ_i} values of ξ_i. Let us suppose the cycle structure of this permutation is

$$\alpha_j^{(\lambda_i)} \quad (j = 1, 2, \ldots, 2^{\lambda_i}),$$

i.e., there are

$$\alpha_j^{(\lambda_i)}$$

cycles of length j in the permutation of the 2^{λ_i} values of ξ_i, induced by the operation of the ith cycle of σ. (This number depends on whether the cycle is an e- or an o-cycle of $N_i\sigma$.) It is clear that a knowledge of the α's suffices to define the cycle structure of the permutation of the s_ν as a function of $N_i\sigma$.

For example, if $K = 2$ and the permutation of the values of ξ_1 has a cycle of length a and the permutation of the values of ξ_2 has a cycle of length b, then the s_ν will have ab/c cycles of length c, where c is the least common multiple of a and b. This may be seen as follows. Without loss of generality we may assume the cycle of length a to be $(12 \ldots a)$ and the cycle of length b to be $(12 \ldots b)$ and $a < b$. We wish to determine the cycle structure of the permutation of the ab symbols (ξ_1, ξ_2) where $\xi_1 = 1, 2, \ldots, a$; $\xi_2 = 1, 2, \ldots, b$. Now $(1, 1)$ will be replaced by $(2, 2)$, $(2, 2)$ by $(3, 3)$, $\ldots$, (a, a) by $(1, a + 1)$, etc. We return to $(1, 1)$ after c steps. Similarly, starting with any of the ab symbols (ξ_1, ξ_2) the original symbol is again obtained after c steps. Since there are only ab symbols, they must be permuted in ab/c cycles each of length c.

These observations for $K = 2$ can be extended to arbitrary K. The following simple calculus for determining the cycle structure of the permutations of the s_ν is then obtained. For each $i = 1, 2, \ldots, K$ form the expression

$$P(\lambda_i) = \sum_{j=1}^{2^{\lambda_i}} \alpha_j^{(\lambda_i)} z_j$$

in the indeterminates z_j. Define multiplication of the z's by

$$z_a z_b = (ab/c) z_c \tag{7}$$

where c is the least common multiple of a and b (an associative law of multiplication when extended to three or more factors). The product

$$P = \prod_1^K P(\lambda_i)$$

can then be expanded in the form $\Sigma \alpha_i z_i$. The α_i are positive integers giving the number of cycles of length i in the permutation of the s_ν induced by $N\sigma$.

There remains only the problem of obtaining the quantities

$$\alpha_j^{(\lambda_i)}.$$

These quantities depend not only on the length λ_i of the cycle in question, but on whether the cycle is an e- or o-cycle. For an e-cycle, $\alpha_j^{(\lambda)}$ can be obtained as follows. Let the numbers from 0 to $2^\lambda - 1$ be written in binary form in natural order in a column. The effect of an e-cycle of length λ on this array may be obtained by removing the left-hand column of the array and writing it in again as the right-hand column. Each original binary number is then doubled modulo $2^\lambda - 1$, and the permutation is easily written; e.g., for $\lambda = 3$ we have (0) (1, 2, 4) (3, 6, 5) (7) and $\alpha_1^{(3)} = 2, \alpha_3^{(3)} = 2$ and all other $\alpha^{(3)}$ are zero. The o-cycle case can be obtained from the e-cycle case by interchanging the zeros and ones in the column of the array in which these symbols alternate from row to row. This corresponds to left-multiplying the permutation obtained in the e-cycle case by the permutation

$$(0, 1)\ (2, 3) \ldots (2^n - 2, 2^n - 1).$$

For $\lambda = 3$, we find $(0, 1, 3, 7, 6, 4)(2, 5)$ whence $\alpha_2^{(3)} = 1$, $\alpha_6^{(3)} = 1$ and all other $\alpha^{(3)}$ are zero. Table I lists the $P(\lambda)$ for e- and o-cycles of length $\lambda = 1, 2, 3, 4, 5, 6$.

TABLE I

λ	$P(\lambda_e)$	$P(\lambda_o)$
1	$2 z_1$	z_2
2	$2 z_1 + z_2$	z_4
3	$2 z_1 + 2 z_3$	$z_2 + z_6$
4	$2 z_1 + z_2 + 3 z_4$	$2 z_8$
5	$2 z_1 + 6 z_5$	$z_2 + 3 z_{10}$
6	$2 z_1 + z_2 + 2 z_3 + 9 z_6$	$z_4 + 5 z_{12}$

It can be shown that the rows of Table I can be extended successively to larger values of λ as follows. For $\lambda = n$ and the case of an e-cycle, the only z's occurring in $P(\lambda_e)$ (e denotes that the cycle of length λ is an e-cycle) are those whose subscripts are integral divisors of n, and every such z occurs. Every such z except z_n has occurred previously in the $P(\lambda_e)$ table and the coefficients of these z's in $P(n_e)$ are taken to be identical with the coefficients in previous occurrences of these z's. Thus

$$P(n_e) = \sum_1^{n-1} \alpha_i z_i + x z_n$$

where only x is unknown; x is then given by

$$x = \left(2^n - \sum_1^{n-1} i \alpha_i\right) \Big/ n.$$

$P(n_o)$ is obtained in a somewhat similar manner. The only z's occurring in $P(n_o)$ are those whose subscripts are integral divisors of $2n$ but are not integral divisors of n, and every such z occurs. All such z's except z_{2n} have occurred previously in the $P(\lambda_o)$ table and the coefficients of these z's in $P(n_o)$ are taken to be identical with the coefficients of these z's in previous occurrences. $P(n_o)$ is thus

$$\sum_1^{2n-1} \alpha_i z_i + x z_n$$

where only x is unknown; x is given by

$$x = \left(2^n - \sum_1^{2n-1} i \alpha_i\right) \Big/ 2n.$$

4. Computational scheme and results of computations. The procedure developed above may be summarized as follows. A partition of n into positive integers,

$$n = \sum_1^K \lambda_i,$$

is written by listing the λ_i in any order. The subscript e or o is added to each λ_i. Each of the distinct possible symbols obtained in this manner specifies a class of O_n and all classes of O_n are obtained. The cycle structure of the permutation of the s_ν induced by all elements of any class C is obtained by forming

$$\bar{P} = \prod_1^K P(\lambda_i) = \sum \alpha_i z_i$$

(using (7)) where the appropriate $P(\lambda)$ are taken from Table I; α_i is the number of cycles of length i in the permutation of the s_ν induced by an element of class C whence the quantities $\lambda_i^{(C)}$ of (5) are obtained. $K(C)$ of (3) is given by $\sum \alpha_i$ and n_C by (6) so that N_n and $N_n^{(m)}$ can then be obtained from (3) and (5).

This computational scheme was used to obtain the following values of N_n:

n	1	2	3	4	5	6
N_n	3	6	22	402	1,228,158	400,507,806,843,728

REFERENCES

1. Harvard Computation Laboratory Staff, *Synthesis of electronic computing and control circuits* (Cambridge, Mass., 1951).
2. F. D. Murnaghan, *The theory of group representations* (Baltimore, 1938).
3. G. Pólya, *Sur les types des propositions composées*, J. Symbolic Logic, *5* (1940), 98-103.
4. C. E. Shannon, *The synthesis of two-terminal switching circuits*, Bell System Tech. J., *28* (1949), 59-98.
5. J. A. Todd, *The groups of symmetries of the regular polytopes*, Proc. Cambridge Phil. Soc., *27* (1931) 212-231.
6. A. Young, *On quantitative substitutional analysis*, Proc. London Math. Soc. (2), *31* (1930), 273-288.

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A REMARK CONCERNING GRAVES' CLOSURE CRITERION

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In a paper recently published in this journal [**1**], R. E. Graves proved a closure criterion for orthonormal sets of functions. A refined form of it may be stated as follows:

THEOREM A. *Let p be a function whose zeros have Lebesgue measure zero, such that for each $x \in (a, b)$, $p \in L_2$ on* $\min(c, x) < t < \max(c, x)$, *where $a < c < b$. (a, b, and c may be infinite.) Let w be measurable, almost everywhere positive, and such that*

$$w(x) \int_c^x |p(t)|^2 dt \in L_1$$

on (a, b). Then for any family $\{\phi_n\}$, orthonormal in (a, b),

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)\, dt \right|^2 w(x)\, dx \leq \int_a^b \left| \int_c^x |p(t)|^2 dt \right| w(x)\, dx,$$

where equality holds if and only if $\{\phi_n\}$ is closed in L_2 on (a, b).

In Graves' version of the theorem, the zeros and discontinuities of p were assumed to have Jordan content zero.

The proof of Theorem A is quite similar to the one given in [**1**]; we merely replace Theorem III of [**1**] by Theorem B below, whose proof is actually simpler than that of Theorem III.

THEOREM B. *If $p \in L_2$ on every compact sub-interval of (a, b) and if $p(t)$ is different from zero almost everywhere on (a, b), then the set of functions of the form*

$$f(t) = \sum_{k=1}^{m} c_k\, p(t)\, \chi_{(a_k, b_k)}(t) \qquad (a < a_k < b_k < b) \tag{1}$$

is dense in L_2 on (a, b).

Here χ_E denotes the characteristic function of the set E.

Proof. Suppose $a < \alpha < \beta < b$, let $g \in L_2$ on (α, β), and suppose $g(t) = 0$ outside (α, β). It suffices to approximate functions of this type in the L_2-norm by functions of the form (1).

We shall do this by showing that the set of functions

$$p(t)\, \chi_{(\alpha, \gamma)}(t) \qquad (\alpha < \gamma < \beta) \tag{2}$$

is complete in L_2 on (α, β). Let $h \in L_2$ on (α, β), and suppose

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$$\int_{\alpha}^{\beta} h(t)p(t)\ \chi_{(\alpha,\gamma)}(t)\,dt = 0 \qquad (\alpha < \gamma \leq \beta),$$

that is,

$$\int_{\alpha}^{\gamma} h(t)p(t)\,dt = 0 \qquad (\alpha < \gamma < \beta).$$

It follows that $h(t)p(t) = 0$ almost everywhere, so that $h(t) = 0$ almost everywhere.

Hence the set of functions (2) is complete in L_2 on (α, β). Theorem B follows, since closure is equivalent to completeness in L_2.

Reference

1. R. E. Graves, *A closure criterion for orthogonal functions*, Can. J. Math., 4 (1952), 198-203.

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BOUNDARY VALUE PROBLEMS ASSOCIATED WITH THE TENSOR LAPLACE EQUATION

G. F. D. DUFF

Introduction. The boundary value problems considered in this paper relate to harmonic p-tensors on Riemannian manifolds with boundary. We study the equation of Beltrami-Laplace

$$\Delta\phi = 0$$

and formulate three boundary value problems which correspond to the Dirichlet, Neumann, and mixed boundary value problems of potential theory. Existence proofs are given by means of the theory of singular integral equations. Essential use is made of the kernal $g_p(x, y)$ for closed manifolds which was introduced by de Rham.

Harmonic fields, which satisfy

$$d\phi = 0, \quad \delta\phi = 0,$$

constitute a distinguished subclass of solutions of the Laplace equation. The harmonic fields are precisely the solutions of the homogeneous second boundary value problem, analogous to the constant solutions of scalar potential theory.

Properties of certain domain functionals are derived from the existence theorems. In order to give a reasonably short proof for the third boundary value problem, we assume the existence of a Green's form for a larger manifold. An eigenvalue problem is examined in the concluding section.

The formal analogy with potential theory is very close throughout. The results may be interpreted as generalizations of the classical existence theorems, as a characterization of certain systems of elliptic partial differential equations, and as an extension of the theory of harmonic integrals on a closed manifold.

1. Manifolds and tensors. Let M be an orientable Riemannian manifold of dimension n and of class C^∞. Let the boundary of M be a regular sub-manifold B of dimension $n - 1$ of M. We suppose that M is finite in the sense that M is covered by a finite number of fundamental coordinate neighbourhoods which are open cubes in suitable local coordinate systems. On M is carried a positive definite metric tensor g_{ij} of the class C^∞.

Associated with M is the double F of M, a closed Riemannian manifold consisting of M, and an oppositely oriented replica $\tilde{M}$ of M, with corresponding boundary points identified. The metric tensor g_{ij} can be extended to F so as to be C^∞ on F, though not necessarily the same at corresponding points of M and of $\tilde{M} = CM$ (complement of M in F) **[2b]**.

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On M there exist skew symmetric covariant tensors

$$\phi_{i_1 \ldots i_p}$$

of rank p, $0 < p < n$, and their associated differential forms [5] of degree p:

$$\phi = \phi_{(i_1 \ldots i_p)}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{1.1}$$

The differentials dx^i anti-commute.

We define the generalized metric tensor,

$$\Gamma_{i_1 \ldots i_p,\, j_1 \ldots j_p} = \begin{vmatrix} g_{i_1 j_1} & \cdots & g_{i_1 j_p} \\ \vdots & & \vdots \\ g_{i_p j_1} & \cdots & g_{i_p j_p} \end{vmatrix}, \tag{1.2}$$

the volume n-tensor density

$$e_{i_1 \ldots i_n} = \Gamma_{i_1 \ldots i_n}{}^{12 \ldots n} \sqrt{\Gamma_{12 \ldots n,\, 12 \ldots n}}\,, \tag{1.3}$$

the covariant derivative

$$D_i\, \phi_{i_1 \ldots i_p} = \frac{\partial}{\partial x^i}\, \phi_{i_1 \ldots i_p} - \sum_{n=1}^{p} \left\{ {}_{i_n\, i}^{\;h} \right\} \phi_{i_1 \ldots i_{n-1} h\, i_{n+1} \ldots i_p}\,, \tag{1.4}$$

the differential

$$(d\phi)_{i_1 \ldots i_{p+1}} = \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)}\, D_j\, \phi_{(j_1 \ldots j_p)}, \tag{1.5}$$

the dual

$$(*\phi)_{j_1 \ldots j_{n-p}} = e_{(i_1 \ldots i_p) j_1 \ldots j_{n-p}}\, \phi^{(i_1 \ldots i_p)}, \tag{1.6}$$

the co-differential

$$(\delta\phi)_{i_1 \ldots i_{p-1}} = -\, \Gamma_{i i_1 \ldots i_{p-1}}{}^{(j_1 \ldots j_p)} g^{ij} D_j \phi_{(j_1 \ldots j_p)}, \tag{1.7}$$

and the Laplacian

$$\begin{aligned}(\Delta\phi)_{i_1 \ldots i_p} = ((d\delta + \delta d)\phi)_{i_1 \ldots i_p} &= -\, D^i D_i \phi_{i_1 \ldots i_p} \\ &+ \sum_{n=1}^{p} \Gamma_{i_1 \ldots i_p}{}^{i(j_1 \ldots j_p)} g^{ij} R^h{}_{j_n j i} \phi_{j_1 \ldots j_{n-1} h j_{n+1} \ldots j_p},\end{aligned} \tag{1.8}$$

where R_{ijkl} is the curvature tensor. The brackets enclosing a set of indices mean that summation is to be effected only over those values which are in increasing order.

If $d\phi = 0$, ϕ is said to be closed; if $\phi = d\chi$, ϕ is derived; if $\delta\phi = 0$, ϕ is coclosed and if $\phi = \delta\chi$, coderived. If $\Delta\phi = 0$, ϕ is said to be a harmonic form and if $d\phi = 0$, $\delta\phi = 0$, ϕ is a harmonic field. Harmonic fields are harmonic forms. Harmonic forms [8] on M are of class C^∞.

For all forms ϕ of degree p, we have

$$d \cdot d\phi = 0, \quad \delta \cdot \delta\phi = 0, \quad **\phi = (-1)^{np+p}\phi, \quad \delta\phi = (-1)^{np+n+1} *d*\phi\,. \tag{1.9}$$

On the boundary B, ϕ defines a tangential p-form $t\phi$ whose components are precisely those components of ϕ with no index n, where x^n is a normal coordinate in some system. The residual part of ϕ is the normal component $n\phi$. The relations

$$*t = n*, \quad *n = t* \tag{1.10}$$

hold on B, and the decomposition $\phi = t\phi + n\phi$ is invariant on B.

We introduce the scalar product of two p-tensors ϕ and ψ:

$$(1.11)\qquad (\phi, \psi) = \int_M \phi \wedge *\psi = \int_M \psi \wedge *\phi.$$

The scalar square $(\phi, \phi) = N(\phi)$ is positive definite. Let

$$(1.12)\qquad D(\phi, \psi) = (d\phi, d\psi) + (\delta\phi, \delta\psi),$$

then $D(\phi) = D(\phi, \phi)$ is the Dirichlet integral associated with the Laplace equation on M. Scalar products and functionals extended over domains other than M will be indicated by subscripts.

The Stokes formula for a $(p + 1)$-dimensional chain C with boundary bC is [5]

$$(1.13)\qquad \int_C d\phi = \int_{bC} \phi,$$

valid for every p-tensor ϕ of class C^1. From Stokes's formula and the formula for the differential of a product follows the formula of Green:

$$(1.14)\qquad (d\phi, \psi) - (\phi, \delta\psi) = \int_B \phi \wedge *\psi.$$

Here ϕ, ψ are of degree p, $p + 1$ respectively. Two other forms of Green's formula, which follow easily from (1.14), will be needed. If ϕ and ψ are of equal degree p, then

$$(1.15)\qquad (d\phi, d\psi) + (\delta\phi, \delta\psi) - (\phi, \Delta\psi) = \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi)$$

and

$$(1.16)\qquad (\Delta\phi, \psi) - (\phi, \Delta\psi) = \int_B (\phi \wedge *d\psi - \psi \wedge *d\phi + \delta\phi \wedge *\psi - \delta\psi \wedge *\phi).$$

The theorem of Hodge [5] for harmonic fields in a closed manifold F states that there exists a unique harmonic field with given periods on the $R_p(F)$ independent (absolute) p-cycles of F. These harmonic fields ω_p^i $(i = 1, \ldots, R_p(F))$ have a reproducing kernel

$$(1.17)\qquad \alpha_p(x, y) = \sum_i \omega_p^i(x)\omega_p^i(y), \quad (\omega_p^i, \omega_p^j)_F = \delta_{ij},$$

in the metric (1.11).

For our existence proofs the de Rham kernel $g_p(x, y)$ of the double F will be needed. This kernel has the following properties, described in [8]:

(a) For every $\phi \in L^2(F)$ (i.e. such that $N_F(\phi) < \infty$), we have

$$(1.18)\qquad \Delta G\phi = G\Delta\phi = \phi - H\phi,$$

where

$$(1.19)\qquad G\phi = (g_p, \phi)_F,$$

and

$$H\phi = (\alpha_p, \phi)_F. \tag{1.20}$$

(b) The kernel satisfies

$$\Delta_x\, g_p(x, y) = -\,\alpha_p(x, y) \qquad x \neq y, \tag{1.21}$$

$$g_p(x, y) = g_p(y, x), \tag{1.22}$$

$$g_p(x, y) \sim \gamma_p(x, y) \qquad x \in N(y), \tag{1.23}$$

where $\gamma_p(x, y)$ is a local fundamental singularity for the Laplace equation $\Delta\phi = 0$. The precise nature of this singularity need not concern us here; we note that

$$g_p(x, y) \sim \gamma_p(x, y) = O(s^{2-n})\,, \tag{1.24}$$

where s is the geodesic distance from x to y. Also, for every form $\phi \in C^1$, we have [**1**]

$$\lim_{\epsilon\to 0} \int_{s=\epsilon} (\phi \wedge *d\gamma - \delta\gamma \wedge *\phi) = \phi. \tag{1.25}$$

(c) The equation

$$\Delta\mu = \phi$$

is solvable in F if and only if $H\phi = 0$, a solution being

$$\mu = G\phi.$$

The orthogonality conditions $HG = GH = 0$ make $G\phi$ unique.

(d) The operator G commutes with d, $*$, and δ. Hence

$$d_x\, g_p(x, y) = \delta_y\, g_{p+1}(x, y) \tag{1.26}$$

and

$$g_{n-p}(x, y) = *_x *_y\, g_p(x, y). \tag{1.27}$$

2. Boundary value problems of the first and second kinds. The boundary value problem of the first kind consists of determining a harmonic form ϕ in M with given tangential and normal boundary components on B. It has been shown that a solution of the problem exists provided that the solution is unique [**2a**]. Uniqueness of the solution holds if the conditions

$$\Delta\phi = 0, \quad t\phi = 0, \quad n\phi = 0 \tag{2.1}$$

imply that ϕ is identically zero. Assume that (2.1) holds; it follows from (1.15) that ϕ is a harmonic field in M:

$$d\phi = 0, \quad \delta\phi = 0.$$

Let x^n be a normal coordinate in a neighbourhood of a given point P of B. From

$$0 = n(d\phi)_{i_1 \dots i_p n} = \Gamma_{i_1 \dots i_p n}{}^{j(j_1 \dots j_p)} D_j \phi_{(j_1 \dots j_p)} = (-1)^p D_n\, \phi_{i_1 \dots i_p}\,,$$

and

$$0 = t(\delta\phi)_{i_1 \ldots i_{p-1}} = \Gamma_{i i_1 \ldots i_{p-1}}{}^{(j_1 \ldots j_p)} D^i \phi_{(j_1 \ldots j_p)} = (-1)^{p-1} D^n \phi_{i_1 \ldots i_{p-1} n}$$

it follows that the first derivatives in the normal direction of all components of ϕ vanish at P. Further differentiation shows that the higher normal derivatives of components of ϕ also vanish at P. It is, therefore, seen that all derivatives of ϕ vanish at P, and hence everywhere on B.

If M is an analytic manifold with analytic metric tensor, it can be shown [**6**] that harmonic forms are analytic. Therefore, the uniqueness holds in this case.

Under certain topological restrictions the uniqueness property holds for C^∞ manifolds. It was proved in [**2a, b**] that there exists a unique harmonic field ϕ with zero tangential boundary value such that either (a) ϕ has given periods on $R_{n-p}(M) = R_p(M, B)$ independent relative p-cycles or (b) $*\phi$ has given periods on $R_{n-p}(M)$ independent absolute p-cycles. Hence the number of independent harmonic fields with zero tangential and normal boundary components cannot exceed the number of p-cycles which are independent both as absolute and as relative cycles. If this number is zero, uniqueness for the first boundary value problem holds.

If uniqueness holds for M, it holds for any sub-manifold M_1 contained in M. For if $\phi \in C^\infty$ is harmonic in M_1 and vanishes on the boundary of M_1, all derivatives of ϕ vanish there and ϕ may be extended to a C^∞ harmonic form in M by defining it to be zero in $M - M_1$. The uniqueness holds if there is a maximum modulus or mean value theorem available; thus it holds for scalars in Euclidean space. Possibly uniqueness holds in general; this has not been proved. The existence proof which is to follow has the advantage that it is valid independently of the uniqueness.

The natural data of the second boundary value problem are $nd\phi$ and $t\delta\phi$, as may be seen from Green's formula. These data satisfy the condition of being self-dual, on account of (1.10). Together there are $\binom{n}{p}$ components, each containing one first normal derivative of a component of ϕ. If $1 < p < n - 1$, certain tangential derivatives also appear. From (1.15) it follows that

$$\Delta\phi = 0, \quad nd\phi = 0, \quad t\delta\phi = 0 \tag{2.2}$$

imply that, in M,

$$d\phi = 0, \quad \delta\phi = 0.$$

The harmonic fields are, therefore, precisely the solutions of the homogeneous boundary value problem of the second kind (zero data); we may refer to them as homogeneous solutions. Likewise, any solution of the non-homogeneous problem is undetermined to the extent of an added harmonic field.

Let ϕ be any harmonic form in M, τ any harmonic field in M. From (1.15) it is seen that

$$\int_B (\tau \wedge *d\phi - \delta\phi \wedge *\tau) = 0. \tag{2.3}$$

This orthogonality condition must be satisfied by the assigned data $nd\phi$ and $t\delta\phi$, for every harmonic field τ in M.

These facts are well known in the scalar boundary value problem, in which the harmonic fields reduce to the constants. The orthogonality condition reduces to the assigned values of the normal derivative having a zero average. In the theory of elasticity [7] the equilibrium equations are the case $n = 3$, $p = 1$ of the slightly more general equation

$$(\delta d + \alpha d\delta)\phi = 0, \qquad \alpha > 0,$$

where α is a constant of the material. The uniqueness properties of this equation are the same as for $\alpha = 1$. The boundary value problem of the first kind in the theory of elasticity corresponds to the assignment of surface displacements, and has a unique solution. The second boundary value problem corresponds to the surface tractions being given; these must satisfy the conditions of rigid body equilibrium for the whole, and the solution is undetermined to the extent of a rigid body motion. Such motions, being irrotational and without divergence, are given precisely by harmonic fields.

3. Potentials. Let $g = g_p(x, y)$ be the de Rham kernel of the double F. We introduce the potentials

$$\mu = \int_B (\rho \wedge *dg - \delta g \wedge *\rho), \tag{3.1}$$

$$\nu = \int_B (g \wedge *d\sigma - \delta\sigma \wedge *g), \tag{3.2}$$

where $t\rho$, $n\rho$, $nd\sigma$, $t\delta\sigma$ are continuous on B. We also suppose that ν satisfies the orthogonality condition (2.3). Both μ and ν are defined and are of class C^∞ in M and in CM. It follows that

$$\Delta\mu = \int_B (\rho \wedge *d\alpha - \delta\alpha \wedge *\rho) = 0,$$

since $d\alpha = 0$, $\delta\alpha = 0$. From (1.16) and (2.3) we have

$$N(\Delta\nu) = -\int_B (\Delta\nu \wedge *d\nu - \delta\nu \wedge *\Delta\nu) = 0,$$

since $d\Delta\nu = 0$, $\delta\Delta\nu = 0$; so that μ and ν are harmonic forms in M and in CM. Thus we are able to use $g_p(x, y)$ as kernel for the potentials in spite of the fact that $g_p(x, y)$ is not a harmonic form.

The potentials (3.1), (3.2) and their derivatives have discontinuities across B, which we now calculate [**2b**]. We note that

$$\Delta(\phi(x), g_p(x, y)) = -(\phi(x), \alpha_p(x, y)) + \begin{cases} \phi(y), & y \in M \\ 0, & y \in CM \end{cases} \tag{3.3}$$

and observe that $\alpha_p(x, y)$ is continuous in F. Also

$$\begin{aligned} \Delta(\phi, g_p) &= \delta d(\phi, g_p) + d\delta(\phi, g_p) \\ &= \delta(\phi, \delta g_{p+1}) + d(\phi, dg_{p-1}) \\ &= d\int_B g_{p-1} \wedge *\phi - \delta\int_B \phi \wedge *g_{p+1} + d(\delta\phi, g_{p-1}) + \delta(d\phi, g_{p+1}). \end{aligned} \tag{3.4}$$

The two last terms on the right-hand side of (3.4) are continuous on B. Since $t\phi$ and $t_*\phi$ may be chosen independently, it follows that the discontinuities of the remaining terms as y crosses B into CM are as follows:

$$(3.5)\quad \begin{aligned} &t_*d\int_B g_{p-1}\wedge *\phi \text{ decreases by } t_*\phi,\ t\delta\int_B \phi\wedge *g_{p+1} \text{ increases by } t\phi,\\ &td\int_B g_{p-1}\wedge *\phi \text{ and } t_*\delta\int_B \phi\wedge *g_{p+1} \text{ continuous.}\end{aligned}$$

For $x \in B$, y in a neighbourhood of x, we have [2b]

$$td\, g_p = O(s^{2-n}),\ s = s(x, y)$$

hence on B we have

$$(3.6)\quad t\delta\int_B g_{p-1}\wedge *\phi \text{ and } t_*d\int_B \phi\wedge *g_{p+1} \text{ continuous.}$$

Noting that

$$\mu = \int_B (\rho\wedge *dg + *\rho\wedge *d*g),$$

$$\nu = \int_B (g\wedge *\, d\sigma + *g\wedge *d*\sigma),$$

and applying these results, we find that $t\mu$, $t_*\mu$, $t_*d\nu$, and $t_*d_*\nu$ have the discontinuities $t\rho$, $t_*\rho$, $-\ t_*d\sigma$, and $-\ t_*d_*\sigma$, respectively, as the argument point passes from M into CM. We conclude that on B,

$$(3.7)\quad \begin{aligned} t\mu &= \tfrac{1}{2}t\rho + t\int_B (\rho\wedge *dg + *\rho\wedge *d*g),\\ t_*\mu &= \tfrac{1}{2}t_*\rho + t_*\int_B (\rho\wedge *dg + *\rho\wedge *d*g),\\ t_*d\nu &= -\ \tfrac{1}{2}t_*d\sigma + t_*d\int_B (g\wedge *d\sigma + *g\wedge *d*\sigma),\\ t_*d_*\nu &= -\ \tfrac{1}{2}t_*d_*\sigma + t_*d_*\int_B (g\wedge *d\sigma + *g\wedge *d*\sigma).\end{aligned}$$

The integrals on the right are to be interpreted as principal values.

By reasoning similar to that used in the scalar potential theory we see that the solution of the first boundary value problem is equivalent to the solution of the equations

$$(3.8)\quad t\mu = t\phi,\quad t_*\mu = t_*\phi$$

where $t\phi$, $t_*\phi$ are the given continuous data of the problem. Similarly, the second boundary value problem is solved by means of the equations

$$(3.9)\quad t_*d\nu = t_*d\phi,\quad t_*d_*\nu = t_*d_*\phi,$$

for given continuous data $t_*d\phi$, $t_*d_*\phi$ on B.

The kernel of the equations (3.8) is

$$\begin{pmatrix} t_x\, t_y *_x d_x\, g_p(x, y), & t_x\, t_y *_x d_x *_x g_p(x, y) \\ t_x\, t_y *_y *_x d_x\, g_p(x, y), & t_x\, t_y *_y *_x d_x *_x g_p(x, y) \end{pmatrix}$$

and the transposed kernel

$$\begin{pmatrix} t_x\, t_y *_y d_y\, g_p(x, y), & t_x\, t_y *_x *_y d_y\, g_p(x, y) \\ t_x\, t_y *_y d_y *_y g_p(x, y), & t_x\, t_y *_x *_y d_y *_y g_p(x, y) \end{pmatrix}$$

is the kernel of the equations (3.9). When $x = y$, the kernels are singular of order $(n - 1)$. Thus (3.8) and (3.9) are systems of singular integral equations.

4. Solution of the integral equations. The condition for the compatibility of (3.8) and (3.9) is that the non-homogeneous terms be orthogonal (in the boundary metric) to every solution of the homogeneous transposed equation [**4**]. In each case the transposed equation arises when we try to solve the boundary value problem of the complementary type for CM.

For the boundary value problem of the first kind we must show that

$$\int_B (\phi \wedge *d\sigma + *\phi \wedge *d*\sigma) = 0, \tag{4.1}$$

where σ is any solution of the equations

$$\begin{aligned} 0 &= \tfrac{1}{2} t*d\sigma + t*d \int_B (g \wedge *d\sigma + *g \wedge *d*\sigma), \\ 0 &= \tfrac{1}{2} t*d*\sigma + t*d* \int_B (g \wedge *d\sigma + *g \wedge *\delta*\sigma). \end{aligned} \tag{4.2}$$

The notations t_-, t_+ will be used to indicate tangential boundary components with limiting values from the interiors of M and CM respectively. The equations (4.2) imply

$$t_+\delta\nu = 0, \quad t_+*\delta\nu = 0. \tag{4.3}$$

We show that ν is a harmonic field in CM. From (1.21) and (3.2) we have

$$\delta d\delta\nu = 0, \quad d\delta d\nu = 0 \qquad \text{in } CM.$$

Now

$$(d\delta\nu, d\delta\nu)_{CM} = (\delta\nu, \delta d\delta\nu)_{CM} + \int_{-B} \delta\nu \wedge *d\delta\nu = 0,$$

so

$$d\delta\nu = 0 \qquad \text{in } CM.$$

Hence

$$(\delta\nu, \delta\nu)_{CM} = (\nu, d\delta\nu)_{CM} - \int_{-B} \delta\nu \wedge *\nu = 0,$$

so

$$\delta\nu = 0 \qquad \text{in } CM.$$

Similarly it is easily shown that

$$d\nu = 0 \qquad \text{in } CM.$$

As the argument point y crosses B, the normal derivatives of components of ν have discontinuities given by (3.5) and (3.6). We find

$$(4.4) \qquad t\_d\nu = 0, \quad t\_{*}d\nu = t{*}d\sigma, \quad t\_\delta\nu = t\delta\sigma, \quad t\_{*}\delta\nu = 0.$$

In M, therefore,

$$(4.5) \qquad \nu = \int_B (g \wedge *d\nu - \delta\nu \wedge *g).$$

It follows that

$$(4.6) \qquad \Delta\nu = -\int_B (\alpha \wedge *d\nu - \delta\nu \wedge *\alpha) = (\alpha, \Delta\nu),$$

since $\alpha = \alpha_p(x, y)$ is a harmonic field. We may assume that the harmonic fields ω_p^i of (1.17) are orthogonal, though not normalized, over M. If any of these fields vanish on B they can be omitted from the kernel $\alpha_p(x, y)$ without effecting the validity of (4.6). We therefore have

$$(\omega_p^i, \omega_p^j) = r_i\,\delta_{ij}, \qquad 0 < r_i < 1,$$

for the remaining ω_p^i. From (4.6),

$$\Delta\nu = \sum x_i\,\omega_p^i, \quad x_i = (\Delta\nu, \omega_p^i)_F,$$

and also

$$\Delta\nu = (\alpha, \sum x_i\,\omega_p^i) = \sum x_i\,r_i\,\omega_p^i.$$

The ω_p^i are linearly independent, so

$$x_i = r_i\,x_i,$$

implying $x_i = 0$ and also $\Delta\nu = 0$ in M.

Since ν is a harmonic form in M, (2.3) holds for ν:

$$(4.7) \qquad \int_B (\tau \wedge *d\nu - \delta\nu \wedge *\tau) = 0 \qquad (d\tau = 0, \delta\tau = 0 \text{ in } M).$$

From Green's formula (1.15) we have

$$D(\nu) = (d\nu, d\nu) + (\delta\nu, \delta\nu) = \int_B (\nu \wedge *d\nu - \delta\nu \wedge *\nu).$$

Supplying for ν the expression (4.5), we have

(4.8)

$$D(\nu) = \int_B \nu{*}d\Big(\int_B g \wedge *d\nu - \delta\nu \wedge *g\Big) - \int_B \delta\Big(\int_B g \wedge *d\nu - \delta\nu \wedge *g\Big) \wedge *\nu.$$

Taking the first term of the four, we denote by B_ϵ the boundary B with a sphere of radius ϵ around the point $y = x$ removed. Then

$$\int_B \nu * d\left(\int_B g_p \wedge *d\nu\right) = \lim_{\epsilon\to 0}\int_B \nu \wedge *d\left(\int_{B_\epsilon} g_p \wedge *d\nu\right)$$
$$= \lim_{\epsilon\to 0}\int_B \nu \wedge *\left(\int_{B_\epsilon} \delta g_{p+1} \wedge *d\nu\right)$$
$$= \lim_{\epsilon\to 0}\int_B \delta\left(\int_{B_\epsilon} \nu \wedge *g_{p+1}\right) \wedge *d\nu$$
$$= \int_B \delta\left(\int_B \nu \wedge *g_{p+1}\right) \wedge *d\nu.$$

The remaining terms may be inverted in a similar way, using (1.26) also. From (1.14),

$$\int_B \nu \wedge *g_{p+1} = -(d\nu, g_{p+1})_{CM} + (\nu, \delta g_{p+1})_{CM} = (\nu, \delta g_{p+1})_{CM},$$

since $d\nu = 0$ in CM. The reversal of sign is due to the orientation of B. The other terms of (4.8) may be transformed in an analogous way, and the result is

$$D(\nu) = \int_B [\delta(\nu, \delta g_{p+1})_{CM} + d(\nu, dg_{p-1})_{CM}] \wedge *d\nu$$
$$- \int_B \delta\nu \wedge *[d(\nu, dg_{p-1})_{CM} + \delta(\nu, \delta g_{p+1})_{CM}]$$
$$= -\int_B [(\nu, \alpha)_{CM} \wedge *d\nu - \delta\nu \wedge *(\nu, \alpha)_{CM}] = 0,$$

in view of (4.7), since $(\nu, \alpha)_{CM}$ is a harmonic field. Hence, finally,

$$d\nu = 0, \quad \delta\nu = 0, \qquad \text{in } M,$$

so that from (4.4) we see that (4.1) is satisfied.

THEOREM I. *Let $t\chi$, $t_*\chi$ be continuous forms on B. There exists a harmonic form ϕ such that on B, $t\phi = t\chi$, $t_*\phi = t_*\chi$.*

The second boundary value problem may be treated in similar fashion. Let $nd\phi$, $t\delta\phi$ be given continuous boundary values satisfying the condition (2.3). Then the integral equations (3.9) are compatible if and only if

$$\int_B (\rho \wedge * d\phi - \delta\phi \wedge *\rho) = 0 \tag{4.9}$$

for every solution ρ of the equations

$$t_+\mu = 0, \quad t_{+*}\mu = 0. \tag{4.10}$$

Let μ, defined by (3.1), satisfy (4.10). In CM, μ is a harmonic form with zero tangential and normal boundary components, hence

$$d\mu = 0, \quad \delta\mu = 0, \qquad \text{in } CM.$$

The discontinuity conditions (3.5) and (3.6) show that

$$t_-\mu = -t\rho, \quad t_{-*}\mu = -t_*\rho. \tag{4.11}$$

Hence in M,

$$\mu = -\int_B (\mu \wedge *dg_p - \delta g_p \wedge *\mu)$$
$$= -\delta \int_B \mu \wedge *g_{p+1} + d \int_B g_{p-1} \wedge *\mu.$$

In CM, $\delta\mu$ is zero, that is

$$\delta d \int_B g_{p-1} \wedge *\mu = 0.$$

However [**2b**],

$$\delta d \int_B g_{p-1} \wedge *\mu = -d\delta \int_B g_{p-1} \wedge *\mu - \int_B \alpha_{p-1} \wedge *\mu$$
$$(4.12) \qquad = -d \int_B dg_{p-2} \wedge *\mu - \int_B \alpha_{p-1} \wedge *\mu$$
$$= -d \int_B g_{p-2} \wedge *\delta\mu - \int_B \alpha_{p-1} \wedge *\mu,$$

Stokes's theorem being applied in the last step. Since

$$tdg = O(s^{-n+2}),$$

the expression (4.12) is continuous across B. Therefore,

$$t_{-}\delta\mu = 0$$

and, dually,

$$n_{-}d\mu = 0.$$

In M, now, μ is a harmonic form with these boundary values; from (1.16) we see that μ is a harmonic field in M.

With the help of (4.11), the orthogonality condition becomes

$$\int_B (\mu \wedge *d\phi - \delta\phi \wedge *\mu) = 0,$$

which is satisfied in view of (2.3), since μ is a harmonic field in M. This proves that the second boundary value problem is solvable.

THEOREM II. *Let $nd\chi$, $t\delta\chi$ be given continuous forms on B, such that*

$$(4.13) \qquad \int_B (\tau \wedge *d\chi - \delta\chi \wedge *\tau) = 0$$

for every harmonic field τ in M. Then there exists in M a harmonic form ϕ such that $nd\phi = nd\chi$, $t\delta\phi = t\delta\chi$.

5. Domain functionals. In this section we assume that the uniqueness condition for the first boundary value problem relative to M is satisfied. It has been shown [**3**] that there exists a fundamental singularity in the large $\gamma_p(x, y)$ for M, in this case. The singularity of $\gamma_p(x, y)$ is of the type (1.24).

Subtraction of a suitable harmonic p-form from γ_p gives the Green's form $G_p(x, y)$ of M, with the properties

$$
\begin{aligned}
&G_p(x, y) \sim \gamma_p(x, y), && x \in N(y),\\
(5.1)\qquad &\Delta_x G_p(x, y) = 0, && x \neq y,\\
&t_x G_p(x, y) = 0, \quad n_x G_p(x, y) = 0,
\end{aligned}
$$

and

$$G_p(x, y) = G_p(y, x),$$

the symmetry being a consequence of Green's formula. The Green's form of M is unique. In terms of $G_p(x, y)$ the solution of the first boundary value problem is given by

$$\phi(y) = \int_B (\phi(x) \wedge *dG_p(x, y) - \delta G_p(x, y) \wedge *\phi(x)). \tag{5.2}$$

Let ϕ be a p-form of class C^2 in M, and vanishing on B. Then the equation

$$\Delta\phi = \rho$$

defines a p-form ρ which is continuous. If ρ is zero, clearly ϕ is zero, so that the correspondence of ϕ to ρ is one-one. With the aid of (1.16) we can express ϕ in terms of ρ by the equation

$$\phi = (G_p, \rho). \tag{5.3}$$

Since Δ commutes with d and $*$, we have the relations

$$*_x *_y G_p(x, y) = G_{n-p}(x, y), \tag{5.4}$$

and

$$\delta_y G_{p+1}(x, y) - d_x G_p(x, y) = \int_B \delta_z G_{p+1}(x, z) \wedge *d_z G_p(y, z), \tag{5.5}$$

similar to the formulae (1.26) and (1.27).

Let ϕ be a p-form with finite Dirichlet integral over M; we have from (1.15) the formula

$$D(\phi, G) = \chi - \phi, \tag{5.6}$$

where

$$\chi = \int_B (\phi \wedge *dG - \delta G \wedge *\phi) \tag{5.7}$$

is a harmonic form with the same boundary values as ϕ.

6. The third boundary value problem. Let

$$A = A_{i_1 \ldots i_p,\, j_1 \ldots j_p}$$

be a double alternating p-tensor, symmetric in the two sets of indices $i_1 \ldots i_p$ and $j_1 \ldots j_p$. We define the p-form $A\phi$ as follows:

$$(A\phi)_{i_1 \ldots i_p} = A_{i_1 \ldots i_p,\, (j_1 \ldots j_p)} \phi^{(j_1 \ldots j_p)}. \tag{6.1}$$

If the invariant

$$A_{(i_1 \dots i_p),\,(j_1 \dots j_p)}\, \phi^{(i_1 \dots i_p)}\, \phi^{(j_1 \dots j_p)} > 0 \tag{6.2}$$

for every non-zero ϕ, A is positive definite. If $A = A(x)$ is positive definite in M, then

$$(\phi, A\phi) > 0 \tag{6.3}$$

for every $\phi = \phi(x)$ not identically zero in M.

The boundary conditions of mixed type which we discuss may be formulated as follows: Let A_p and A_{n-p} be two double p-tensors, symmetric and positive definite on the $(n-1)$-dimensional boundary B of M. We require as boundary conditions

$$\begin{aligned} t{*}d\phi + *_B(A_p\, t\phi) &= \chi_{n-p-1}, \\ t{*}d{*}\phi + *_B(A_{n-p} t{*}\phi) &= \chi_{p-1}, \end{aligned} \tag{6.4}$$

where χ_{p-1}, χ_{n-p-1} are continuous forms, of the degrees indicated, on B.

The uniqueness condition (2.1) will be assumed to hold for M. A harmonic form ϕ which satisfies the homogenous conditions (6.4) is then identically zero. For, from (1.15) we find

$$\begin{aligned} D(\phi) &= \int_B (\phi \wedge *d\phi + *\phi \wedge *d*\phi) \\ &= -\int_B (t\phi \wedge *_B(A_p t\phi) + t{*}\phi \wedge *_B(A_{n-p} t{*}\phi)) \\ &= -(t\phi, A_p t\phi)_B - (t{*}\phi, A_{n-p} t{*}\phi)_B \leq 0. \end{aligned} \tag{6.5}$$

Hence both sides of (6.5) must vanish, so that $t\phi = 0$, $n\phi = 0$ on B. The uniqueness condition now implies that ϕ is zero in M. If, therefore, a harmonic form satisfies the non-homogeneous conditions (6.4), it is unique.

For the existence proof we shall assume that M is contained in the interior of a manifold M' having a Green's form (5.1). The difference $M' - M$ may be chosen to be a product of B with an interval, so that the uniqueness property holds for $M' - M$.

The potential

$$\nu = \int_B (G_p \wedge *d\sigma + *G_p \wedge *d*\sigma) \tag{6.6}$$

is a harmonic form in M and in $M' - M$. The analysis of the discontinuities on B carries over unchanged from §3. We suppose that ν satisfies (6.4) and find the integral equations (with principal values understood, as before)

$$\begin{aligned} \chi_{n-p-1} = -\tfrac{1}{2}t{*}d\sigma &+ \int_B (t{*}d_y G + *_B A_p t_y G) \wedge *d\sigma \\ &+ \int_B (t{*}d_y *_z G + *_B A_p t_y *_z G) \wedge *d*\sigma, \end{aligned} \tag{6.7}$$

$$\chi_{p-1} = -\tfrac{1}{2} t{*}d{*}\sigma + \int_B t{*}d{*}_y G + {*}_B A_{n-p} t{*}_y G) \wedge {*}d\sigma$$
$$+ \int_B (t{*}d{*}_y {*}_z G + {*}_B A_{n-p} t{*}_y {*}_z G) \wedge {*}d{*}\sigma.$$

Define the harmonic form

$$(6.8) \qquad \chi = \int_B \rho \wedge {*}({*}dG + {*}_B A_p\, tG) + {*}\rho \wedge ({*}d{*}G + {*}_B A_{n-p}\, t{*}G),$$

then the homogeneous transposed equation associated with (6.7) arises from solving

$$(6.9) \qquad t_+\chi = 0, \quad t_+{*}\chi = 0,$$

where the $+$ sign denotes values from $M' - M$. Let ρ be any solution of (6.9); since χ vanishes on the boundary of M' and on B, it follows that χ is zero in $M' - M$.

From the interior of M, χ has the boundary values

$$(6.10) \qquad t_-\chi = -t\rho, \quad t_-{*}\chi = -t{*}\rho\,;$$

as follows from (3.5) and (3.6). Noting (4.12) and (5.5) as well, we find that

$$(6.11) \qquad t_-{*}d\chi = {*}_B A_p\, t\rho\,; \quad t_-{*}d{*}\chi = {*}_B A_{n-p}\, t{*}\rho.$$

It follows that χ satisfies the homogeneous boundary conditions (6.4) from the interior of M, hence χ is zero in M. This implies that $t\rho = 0$, $t{*}\rho = 0$, and proves that the equations (6.7) have a unique solution.

THEOREM III. *Let M be interior to a manifold M' for which the uniqueness condition* (2.1) *holds; then the boundary value problem* (6.4) *is uniquely solvable for given continuous* χ_{p-1}, χ_{n-p-1} *on* B.

7. Eigenvalues. Let M be a finite manifold, with non-zero boundary B, which satisfies the conditions of §1, and for which the uniqueness condition (2.1) holds. Consider the equation

$$(7.1) \qquad \Delta\phi = \lambda\phi$$

with the boundary condition

$$(7.2) \qquad t\phi = 0, \quad n\phi = 0.$$

From (1.15) it follows easily that (7.1) has no negative eigenvalues, and the uniqueness condition shows that zero is not an eigenvalue. The integral equation which corresponds to (7.1) and (7.2) is

$$(7.3) \qquad \phi = \lambda(G, \phi).$$

The iterated kernels of sufficiently high order of this equation are continuous. In view of (5.3), it follows that there exists a set of eigenvalues $\lambda_n > 0$, and eigenforms ϕ_n, complete in the L^2 space of p-forms which satisfy (7.2). The ϕ_n may be chosen to satisfy

$$(\phi_n, \phi_m) = \delta_{mn}, \quad D(\phi_n, \phi_m) = \lambda_n \delta_{mn}. \tag{7.4}$$

Suppose now that M is a closed manifold ($B = 0$). In view of Hodge's theorem, zero is an eigenvalue of multiplicity $R_p(M)$; and all other eigenvalues are positive. Clearly the harmonic component of any solution of (7.1) with $\lambda \neq 0$ is zero. The integral equation is now

$$\phi = \lambda(g, \phi). \tag{7.5}$$

There exists a set of eigenforms complete in the L^2 space of p-forms on M with zero harmonic component, in view of (1.18). These, together with the Hodge forms, are complete in $L^2(M)$.

References

1. P. Bidal and G. de Rham, *Les formes differentielles harmoniques*, Comment. Math. Helv., *19* (1946), 1–49.
2. G. F. D. Duff and D. C. Spencer,
   (a) *Harmonic tensors on manifolds with boundary*, Proc. Nat. Acad. Sci., *37* (1951), 614–619.
   (b) *Harmonic tensors on Riemannian manifolds*, Ann. Math., *56* (1952), 128-156.
3. P. R. Garabedian and D. C. Spencer, *A complex tensor calculus for Kähler manifolds*, to appear in Acta Math.
4. G. Giraud, *Equations et systèmes d'équations où figurent des valeurs principales d'intégrales*, C. R. Acad. Sci., Paris, *204* (1937), 628–630.
5. W. V. D. Hodge, *The theory and applications of harmonic integrals* (Cambridge, 1941).
6. K. Kodaira, *Harmonic fields in Riemannian manifolds*, Ann. Math., *50* (1949), 587–665.
7. A. E. H. Love, *The mathematical theory of elasticity* (Cambridge, 1927).
8. G. de Rham and K. Kodaira, *Harmonic integrals*, Mimeographed lectures, Institute for Advanced Study, 1950.

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EXTENSION OF COVERINGS, OF PSEUDOMETRICS, AND OF LINEAR-SPACE-VALUED MAPPINGS

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1. Introduction. Let A be a closed subset of a topological space. We show that the following three conditions are equivalent.

1.1 *Any countable neighbourhood-finite open covering of* A (the topological terms referring to the relative topology of A) *has a refinement which can be extended to be a countable neighbourhood-finite open covering of* X.

1.2 *Any separable pseudometric on* A *can be extended to a separable pseudometric on* X.

1.3 *Any mapping of* A *into a separable closed convex subset* S *of a Banach space* B *can be extended to* X, *keeping the values still in* S.

The three conditions obtained by omitting "separable" and "countable" throughout are also shown to be equivalent.

Thereupon we show that 1.1 to 1.3 are always true if X is normal. This is a true generalization of Tietze's extension theorem.

Without the words "countable," "separable," the conditions hold when X is paracompact (and thus normal). We do not know if normality suffices in the general case.

We then take a closer look at the fact that S in 1.3 is restricted to be *closed*. We find that this restriction may be relaxed (that is, the word "closed" omitted) when X is normal and B is one-dimensional. On the other hand, we refer to a paracompact space X and a mapping f on A with values in the *plane* whose extension to X always requires enlargement of the convex hull of the values.

2. Some properties equivalent to extendability. A *pseudometric* r defined on a topological space X is a real-valued function of two variables in X such that $r(x, x) = 0$, $r(x, y) \geqslant 0$, $r(x, z) \leqslant r(x, y) + r(y, z)$ and such that for each y the set of all x such that $r(x, y) < c$ is open. The latter set is an *r-sphere of radius c about y*. We shall call r *γ-separable* if γ is an infinite cardinal number and if there is a subset G of power (cardinal number) not greater than γ which has a non-void intersection with every r-sphere of positive radius.

We shall abbreviate "neighbourhood" by "nbd." When we talk about a subset A of a space it is always implied that this set A is closed. Now consider the following statements about such a set A.

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2.1 *Let C be a nbd-finite* [cf. **7**] *open covering of A, where the power of C does not exceed γ. Then there is a refinement C' of C which can be extended to a nbd-finite open covering C'' of X, where the power of C'' also does not exceed γ.*

2.2 *Any γ-separable pseudometric r defined on A can be extended to a γ-separable pseudometric r' of X.*

2.3 *Let f be a continuous mapping of A into a metrizable γ-separable subset S of a convex topological linear space L. Unless there is a real-valued continuous function g which vanishes precisely on A, assume that S is complete in some one of its metrics. Then f can be extended to X so as to map X into S.*

Now the relation between these is the following.

2.4 THEOREM. *The statements* 2.1, 2.2, 2.3 *are equivalent.*

We shall prove first that each of 2.1 and 2.3 implies 2.2, and then we shall prove that 2.2 implies the others. It is to be noted that in the proof that 2.3 implies 2.2 (and thus 2.1), 2.3 is utilized only for the case in which L is a Banach space.

Let q be a γ-separable pseudometric on A. For each positive integer n cover A by q-spheres of radius $2^{-(n+2)}$. Now the theorem in [**7**] clearly holds for pseudometrics as well as for metrics, so that there is a nbd-finite refinement C, and C can be supposed to have not more than γ members. Let C_1 be the covering of X provided by 2.1. By repeated application of a corollary in [**6**, p. 17] we can obtain a normal sequence [**8**, p. 46] of coverings U_n ($n = 1, 2, \ldots$) of which U is the first covering U_1. By [**8**, p. 51] there is an "associated pseudometric" r_m such that [**8**, p. 51, 7.5] $r_m(x, y) \leqslant 2^{-(n+2)}$ implies x and y are both in some common element of U_n, and thus that $q(x, y) < 2^{-(m+2)}$. Moreover, $r_m(x, y)$ is bounded by 1. Let $r(x, y) = \Sigma_m 2^{-m} r_m(x, y)$. Suppose $r(x, y) < 2^{-k}$ for some positive integer k. Then $r_m(x, y) < 2^{m-k}$ for all m. Let $m = k + 3$ and $n = 1$. Then $r_m(x, y) < 2^{-3}$ and so $q(x, y) < 2^{-(k+5)}$. Let X/r be the class of sets of r-diameter zero, metrized with r. The set A corresponds to a set B in X/r, and from the dominance of q by r shows that q can be defined continuously on B and even to the closure of B. By [**1**, 3.5] q can be extended to X/r and thus to X, as desired. Since for every positive e there is a covering of X by not more than γ q-spheres of radius less than e, the pseudometric q is γ-separable on X.

Now assume 2.3 and let q be given as before. Let $C(A, q)$ be that class of bounded real-valued functions f on A, such that for every positive e there is a positive d such that if $q(x, y) < d$ then $|f(x) - f(y)| < e$. It is not hard to see that $C(A, q)$ is a γ-Lindelöf Banach space with the norm $\|f\| = \sup_{x \in A} |f(x)|$. The mapping ϕ, where

$$\phi(a)(x) = q(a, x) - q(a_0, x), \qquad a_0 \text{ fixed in } A,$$

sends A continuously into $C(A, q)$. This mapping can presumably be extended

to X. Then $q(x, y) = \| \phi(x) - \phi(y) \|$ extends q to all of X, as desired. The γ-separable property is easily seen to hold for the extension.

For both parts of the following second half of the proof of 2.4, let 2.2 hold. First 2.1 will be established. Let U be a nbd-finite open covering for A, with not more than γ sets. There exists a Δ-refinement U_2 of U [6], and it can be seen from Morita's proof that U_2 need have no more than γ sets. Iteration, and taking every other covering so constructed gives a normal sequence beginning with U. Then the associated [7] pseudometric r being γ-separable, can be extended to all of X, so as to be γ-separable. Its relation to U is such that the spheres of r-radius less than $\frac{1}{4}$ are (on A) a refinement of U. By the obvious extension of A. H. Stone's theorem to the case of a pseudometric space, the covering by the spheres of r-radius $\frac{1}{4}$ has a nbd-finite refinement which may be assumed to have not more than γ sets. Thus 2.1 is derived from 2.2.

If S is complete, then the deduction of 2.3 from 2.2 is given by the proof of 4.1 in [1] except for the detail about γ, which can be supplied by the reader. In the proof in question, the completeness of S is used in extending f from A to A [1]. Now if there is a continuous real-valued g which vanishes precisely on A (in a normal space, this is equivalent to A's being the intersection of a countable family of open sets), then q in the proof of 4.1 in the paper just referred to can be replaced by q', $q'(x, y) = q(x, y) + | g(x) - g(y) |$, whereupon $A_0 = A$ and no completeness is needed. Thus 2.4 is proved.

We shall see later that the requirement that either some g vanish precisely on A or that S be complete cannot be ignored.

If A is a closed subset of a topological space such that any and hence all of 2.1, 2.2, and 2.3 hold, we shall say that A is *γ-normally embedded* in X. An earlier result [1, 3.5 and 4.1] can thus be formulated as follows [8; 7]:

2.5 THEOREM. *In a fully normal space X every closed subset A is γ-normally embedded in X for every infinite cardinal number γ.*

3. Countable normal imbedding. In this section we shall show that a (closed) set A in a normal space is always $\aleph_0$-normally imbedded. If it were known that every normal space was "countably paracompact" [3] the proof would be particularly easy, via 2.1 as follows: extend each V in C to X, adjoin the complement, and obtain a nbd-finite refinement of this countable covering as promised by the axiom of countable paracompactness. However, it is possible to circumvent this assumption.

3.1 THEOREM. *If A lies in a normal space, then A is countably, normally imbedded, that is,* 2.1 *to* 2.4 *hold with $\gamma = \aleph_0$.*

Proof. We shall establish 2.1. Let $C = \{U_1, U_2, \dots\}$ be a countable nbd-finite open covering for A. Obtain closed sets $A_n \subset U_n$ which cover A [5]. For each n construct a continuous real-valued f_n such that $0 \leqslant f_n(a) \leqslant 1$, $f_n(a) = 0$ for a in A_n, $f_n(a) = 0$ for a not in U_n. Extend each f_n to all of X,

without increase of bounds, and call the extension g_n. Let P be the topological product of countably many intervals; then P is a (compact) metric space. We can map X into P as follows:

$$x \to g(x) = \{g_1(x), g_2(x), \dots\} \in P.$$

Let V_n in P be the set of all elements $\{t_1, t_2, \dots\}$ in P for which t_n is positive. By the theorem in [**7**] or even the earlier result in [**2**] there is a nbd-finite system of open sets W_n in P whose union is the same as that of the V_n. That this system can be assumed countable is obvious since P is separable. The inverse images G_n under g of these W_n form nbd-finite system S which covers A. Adjoining the complement G_0 of A gives a covering of X. Now let $a \in G_n, a \in A$. Then $g(a) \in W_n \subset V_m$ for some m depending only on n. Therefore $g_m(a) > 0$ or $a \in U_m$. Hence S is a refinement of C. Thus 2.1 is proved for $\gamma = \aleph_0$, as desired.

This result generalizes an earlier result [**1**, 4.3]. Of course, it generalizes Tietze's extension theorem inasmuch as it says that a continuous function on A in a normal space X, with values in a separable Banach space (for example) can be extended without increasing the closed convex hull of the range of values.

In an abstract of this paper (Bull. Amer. Math. Soc., *57* (1951), 487) there was announced an example of a normal space in which a closed set A was not $\aleph_1$-normally imbedded, but the argument now appears to be fallacious and the question remains open.

4. Extension of mappings into non-closed convex sets. There is a refinement of Tietze's extension theorem which is not widely known. The theorem of Tietze says that if A is closed in a normal space X, and f maps A into a *closed* interval K, then f can be extended to X with values in K. But what if the interval K is open or half open? The following theorem shows that the extension is still possible without bringing in values outside K.

4.1 THEOREM. *Let A be a closed set in a normal space X. Let K be a convex set with non-void interior in a topological linear space L. Let T be*

4.11 *an F_σ-set*

contained in the frontier of K. Let f be a mapping of X into K but such that $f(A)$ avoids T. Then f can be so deformed at points not in A so as to have values in K but to avoid T altogether.

Proof. The inverse image S of T under f is an F_σ disjoint from A. We can therefore construct a real-valued mapping g vanishing on A with $0 \leq g \leq 1$ but $g(x) > 0$ for x in S. Select a point k in the interior of K. Define $h(t) = (1 - tg)f + tgk$ for real t, $0 \leq t \leq 1$. Then $h(t)$ agrees with f on A, $h(0) = f$, and $h(1)$ avoids T as desired.

The refinement of Tietze's theorem is obtained by letting T be the class of such end points of K as are not members of K.

A similar result can be obtained when A is a G_δ-set.

4.2 THEOREM. *Let A, X, K, L, T, and f be as in* 4.1, *ignoring* 4.11. *Suppose A is a G_δ-set. Then the conclusion of* 4.1 *holds.*

The proof is exactly as before, because now there exists a function g vanishing exactly on A.

It is of interest to point out that when the dimension of L is greater than 1, the omission of 4.11 would destroy the validity of 4.1. A corresponding statement holds for 4.2. It is possible to construct a normal (in fact, fully normal) space X with a closed subset A which can be mapped into a convex set K in the plane by a mapping which cannot be extended to X without exceeding K. For the set K take the "Example 5.1" of [**4**, p. 381]. Hanner tells how to imbed K as a closed set into a space which is normal and which can be used as the above X. For the sake of brevity, we omit the proof of the full normality of X.

5. Paracompactness not strictly necessary. When X is paracompact then each A is γ-normally imbedded for each γ (see 2.5). It is interesting to observe that the converse is not true. The space that shows this is the space T_{ω_1} of all countable ordinal numbers which is not paracompact [**2**] since it is not compact, but being Fréchet compact, permits no infinite family of sets to be nbd-finite. However, it satisfies 2.3 and in fact any mapping on a subset A can be extended. Any A is in fact a retract. We show how to construct, for A (closed in T_{ω_1}), a mapping f on T_{ω_1} to A such that $f(a) = a$ for a in A. For x in A define $f(x)$ to be the least element of A greater than x, but if there is no such element let $f(x)$ be the greatest element of A which must now exist because A is closed. This f is easily seen to be continuous.

REFERENCES

1. R. Arens, *Extensions of functions on fully normal spaces*, Pacific J. Math., *2* (1952), 11–22.
2. J. Dieudonné, *Une généralization des espaces compacts*, J. Math. Pure Appl., *23* (1944), 65–76.
3. C. H. Dowker, *On countably paracompact spaces*, Can. J. Math., *3* (1951), 219–224.
4. O. Hanner, *Solid spaces and absolute retracts*, Ark. Mat. *1* (1951), 375–382.
5. S. Lefschetz, *Algebraic topology* (Princeton, 1942).
6. K. Morita, *On the dimension of normal spaces* II, J. Math. Soc. Japan, *2* (1950), 16–33.
7. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc., *54* (1948), 677–688.
8. J. W. Tukey, *Convergence and uniformity in topology* (Princeton, 1951).

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REGULAR SURFACES OF GENUS TWO: PART I

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The present paper is a sequel to one I published lately (**3**) on regular surfaces of genus 3, and like it, is intended to fill up some of the gaps in our detailed knowledge of the regular surfaces of moderately low genus $p = p_g = p_a$ and linear genus $p^{(1)} = n + 1 > 1$. (The surfaces for which $p^{(1)} = 1$ form a rather separate field of study on which a good deal of work has been done, and I shall not consider them.) There is for $p = 2$ no canonical model, since the canonical system is only a pencil; but there is in general a unique bicanonical model, about which we shall find that something can be said. The problem, like most similar problems, increases sharply in difficulty with increase of the linear genus, and it is only for the first few values of $p^{(1)}$ that anything like a complete classification of the surfaces in question can be obtained. On account of the length of the work, I am publishing here some general results, and the detailed study of the cases in which $p^{(1)} < 4$, and shall hope to extend the classification to the cases $p^{(1)} = 5, 6$, with some examples of surfaces for $p^{(1)} = 7$, in a subsequent paper.

The notation $[r]$ will be used throughout for the r-dimensional linear space.

1. Generalities. We may begin with one very general result:

THEOREM 1.1 *Every regular surface of genus* 2 *and linear genus* $n + 1$, *whose canonical system is irreducible, has as bicanonical model a surface* F^{4n} *of order* $4n$ *in* $[n + 2]$, *which lies on a quadric cone* Γ_{n+1}^2 *with* $[n - 1]$ *vertex* Ω_{n-1}, i.e., the cone generated by the $[n]$'s joining Ω_{n-1} to the points of a conic in a plane skew to Ω_{n-1}; *and the canonical pencil is traced on* F^{4n} *by the generating* $[n]$'*s of* Γ_{n+1}^2.

For the grade of the bicanonical system is four times that of the canonical, i.e., $4n$; and its freedom is $P - 1$, where by a known formula (**7**, p. 159)

$$P = p^{(1)} + p = n + 3.$$

This gives the order and ambient dimensions of the bicanonical model. Now the bicanonical system is adjoint to the canonical, i.e., on the bicanonical model each curve of the canonical pencil appears as its own canonical model, a curve C^{2n} in $[n]$. The canonical pencil $|C^{2n}|$ has n base points $A_1, \ldots, A_n$, which form a semicanonical set on the general curve of the pencil, i.e., a set in which the curve is touched by an $[n - 1]$ in its ambient $[n]$. Since, moreover, on a regular surface the characteristic series of a complete system, in particular of the canonical system, is complete, the set $A_1, \ldots, A_n$ forms a complete series on each C^{2n}, which means that their join is an $[n - 1]$ and not less, since every $[n - 1]$

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through them must trace on the curve an equivalent set. Their join is thus the unique $[n-1]$, Ω_{n-1}, which touches every C^{2n} in these points; Ω_{n-1} lies in the ambient $[n]$'s of all the curves C^n, which accordingly generate a cone of $n+1$ dimensions, and of order 2, since there are hyperplane sections of F^{4n} consisting of two curves C^{2n}, and hence hyperplane sections of the cone consisting of two of the $[n]$'s; i.e., the cone is the quadric cone Γ_{n+1}^2.

We have a more definite but restricted result in the following:

THEOREM 1.2 *Let G_3^{2n} be a $2n$-ic threefold in* $[n+2]$, *whose general* $[n]$ *section is a canonical curve of genus* $n+1$, *so that its general* $[n+1]$ (i.e., hyperplane) *section, is a surface G_2^{2n} of genus* 1, *whose unique canonical curve is the null curve* (**7**, p. 247); *and let G_3^{2n}, Γ_{n+1}^2 have simple contact along a normal rational n-ic curve K^n lying in a generating* $[n]$ *X_n of Γ_{n+1}^2; then their surface of intersection, F^{4n}, is the bicanonical model of a surface of genera $p = 2$, $p^{(1)} = n+1$.*

(By simple contact is intended that the general point of K^n is double on F^{4n}, though simple on G_3^{2n} and Γ_{n+1}^2; this requires that the tangent [3] to G_3^{2n} at such a point is the join of the tangent planes to the two sheets of F^{4n}, and lies in Y_{n+1}, the tangent hyperplane to Γ_{n+1}^2 at all points of X_n.)

For the general hyperplane section f^{4n} of F^{4n} is a quadric section of a hyperplane section G_2^{2n} of G_3^{2n}; and it has n double points, at its intersections with K^n. On G_2^{2n} every complete linear system, in particular that of quadric sections, is adjoint to itself; the canonical series on f^{4n} is thus traced by quadrics through its n double points; and thus the system adjoint to the hyperplanes sections of F^{4n} is traced on it residually by quadrics through the double curve K^n. The difference between this adjoint system and the hyperplane sections themselves, i.e., the canonical system, is accordingly traced residually on F^{4n} by the pencil of hyperplanes through K^n, i.e., through X_n. But the generating $[n]$'s of Γ_{n+1}^2 trace on G_3^{2n} and hence on F^{4n}, canonical curves C^{2n} of genus $n+1$, any two of which are a hyperplane section of F^{4n}; they thus form a pencil on F^{4n}, traced residually by the pencil of hyperplanes through any one of them. But K^n, counted as double, is one of these curves, being the complete intersection of X_n with G_3^{2n}; the canonical system on F^{4n} is thus just the pencil $|C^{2n}|$, from which the theorem at once follows.

As a partial converse to this we have the result:

THEOREM 1.3 *If the bicanonical model F^{4n} of a regular surface of genera $p = 2$, $p^{(1)} = n+1$, with irreducible canonical pencil, is the complete intersection of Γ_{n+1}^2 with some threefold, the latter can only be the threefold G_3^{2n} whose* $[n]$ *sections are canonical curves of genus* $n+1$, *touching Γ_{n+1}^2 along a normal rational curve K^n in a generating* $[n]$, *X_n, of Γ_{n+1}^2.*

For the threefold is of order $2n$, and has at least some $[n]$ sections which are canonical curves of genus $n+1$; thus its general $[n]$ section cannot be of lower genus than this, nor (since those which we know to be of genus $n+1$ are without singularities) can it be of higher genus; and since its ambient is $[n]$, it is the

canonical curve of this genus. The threefold is thus G_3^{2n}. Now the adjoint system to the hyperplane sections $|2C^{2n}|$ of F^{4n} is $|3C^{2n}|$, and is traced residually on the surface by quadrics through any one curve C^{2n}, i.e., through any one generating $[n]$ of Γ_{n+1}^2; and by a known formula (**7**, p. 61) the genus of the general hyperplane section f^{4n} is $3n + 1$; but f^{4n} is a quadric section of a hyperplane section G_2^{2n} of G_3^{2n}, and on G_2^{2n} belongs to a complete linear system of genus $4n + 1$, adjoint to itself; to be of genus $3n + 1$, f^{4n} must have n double points, and its canonical series will be traced residually by quadrics through these, which must consequently be its complete intersection with a certain generating $[n]$, X_n, of Γ_{n+1}^2. By varying the secant hyperplane so as to keep one of these double points fixed, we see that X_n is the same generating $[n]$ of Γ_{n+1}^2 for all hyperplane sections of F^{4n}; F^{4n} has accordingly a double curve K^n which is its complete intersection with X_n, and is thus likewise the complete intersection of X_n with G_3^{2n}. This curve is rational, since its ambient X_n is of dimensions equal to the order of the curve.

We may devote a few lines to some properties of the surface constructed in these two theorems. The base points of the pencil $|C^{2n}|$ are of course the intersections of G_3^{2n} with Ω_{n-1}; and as such a point A_i is on the double curve K^n, the tangent [3] to G_3^{2n} at A_i lies in Y_{n+1}, and thus meets Ω_{n-1} in a line, the tangent line at A_i to the section of G_3^{2n} by a general $[n]$ through Ω_{n-1}, and thus to the general curve C^{2n}. Thus as we expect, Ω_{n-1} touches each C^{2n} in each of the points $A_1, \ldots, A_n$, i.e., these are a semicanonical set on C^{2n}. Since, moreover, K^n counted twice is a curve of the pencil $|C^{2n}|$, it is the image (on a model of the surface without singularities or exceptional curves) of a hyperelliptic curve of genus $n + 1$, the pairs of whose unique involution are neutral for the bicanonical system, and the united (or Jacobian) points of this involution, $2n + 4$ in number, correspond to pinch points of F^{4n}. Further, since $A_1, \ldots, A_n$ are a semicanonical set on this curve also, they are n of the pinch points. This explains how it is that the curves C^{2n} all have the same tangent in a point A_i, though on a non-singular model the corresponding point is only a simple base point of the pencil. For a curve on the non-singular model which passes simply through the coincident neutral pair corresponding to A_i, corresponds to a curve on F^{4n} with a cusp at A_i, whereas a curve passing through only the first point of the pair (not touching the hyperelliptic curve) corresponds to a curve on F^{4n} passing simply through A_i and touching a fixed line, the principal tangent; and all curves on a surface which pass simply through a pinch point must touch this line. Thus we may say that the curves C^{2n}, though they all touch each other *in space* at A_i, have only a simple intersection there *on the surface*. In this connexion we have the following simple remark:

THEOREM 1.4 *On the bicanonical model F^{4n} of any regular surface of genera $p = 2$, $p^{(1)} = n + 1$, whose canonical pencil is irreducible and non-hyperelliptic, every base point A_i of this pencil is either a simple point, in which case the tangent plane to F^{4n} in this point lies in Ω_{n-1}, or a pinch point, in which case the principal tangent lies in Ω_{n-1}, and is the tangent to all curves of the canonical pencil.*

For since there are hyperplane sections of F^{4n} consisting of two curves C^{2n}, each of which passes simply through A_i, the multiplicity of A_i cannot exceed 2; and since on a model without singularities or exceptional curves A_i corresponds to a single point and not a curve, if it is a double point it can only be a pinch point; I have shown elsewhere that the only types of double point mapped by points only on a non-singular model are the pinch point and the intersection of two simple sheets. (Du Val (**2**). The result is not stated explicitly, but follows from the results of §11 as the multiplicity of the point is $\Sigma\sigma_i^2$, which can only equal 2 if $\sigma_1 = \sigma_2 = 1$.)

It must be understood for the purposes of this theorem that by saying that A_i is a pinch point we only mean that it is the image of a coincident neutral pair for the bicanonical system on a non-singular model, not that the pinch point necessarily occurs in the course of a double curve, i.e., that the coincident neutral pair is one of a singly infinite system of neutral pairs. On the surface we have just constructed this is so, and it is so on any surface in [3]; but in higher space, just as we can have an isolated intersection of two simple sheets which is not part of any multiple curve, so we can perfectly well have a pinch point which is not on any multiple curve of the surface.

We may next observe that

THEOREM 1.5 *Let F^{4n} be the bicanonical model of a surface of genera $p = 2$, $p^{(1)} = n + 1 > 3$, with irreducible and non-hyperelliptic canonical pencil $|C^{2n}|$; then if all the base points of this pencil are pinch points, the totality of quadrics on which F^{4n} lies is a linear system of freedom $\frac{1}{2}(n-1)(n-2)$, of which any subsystem of freedom $\frac{1}{2}n(n-3)$ (not containing Γ_{n+1}^2) traces on the general generating $[n]$ of Γ_{n+1}^2 the complete system of quadrics through the corresponding curve C^{2n}.*

This depends on the lemma that the sections by any hyperplane of the quadrics through an irreducible normal manifold M are the complete system of quadrics in the hyperplane through the section of M. This must certainly be well known, but as I am not aware where it is to be found I give a brief proof here. Let H be a hyperplane whose intersection with M is M_H, and let Q_H be any quadric in H passing through M_H. Let Q be any quadric in the whole space whose intersection with H is Q_H, then its intersection with M consists of M_H together with something coresidual to a hyperplane section, and since M is normal this can only be the section M_K by some hyperplane K. The pencil of quadrics determined by Q and the hyperplane pair H, K all have the same complete intersection M_H, M_K with M, and all have the same intersection Q_H with H. But one quadric of this pencil can be found to pass through a further point of M (not on M_H or M_K), and hence, since M is irreducible, to contain the whole of M; thus Q_H is the section by H of a quadric through M.

We return to the proof of the theorem. The curves $|C^{2n}|$ all touch the principal tangent to the surface in each base point A_i, i.e., they all trace on Ω_{n-1} the same set of $2n$ points (coinciding by pairs). As C^{2n} is normal and irreducible, the sections by Ω_{n-1} of the quadrics Q_{n-1}^2 through C^{2n} in its ambient $[n]$, a linear

system of freedom $\frac{1}{2}n(n-3)$, are the complete system of quadrics Q_{n-2}^2 in Ω_{n-1} passing through these $2n$ points, i.e., touching the principal tangent to F^{4n} in each point A_i. Every quadric Q_{n-2}^2 of this system thus determines in each generating $[n]$ of Γ_{n+1}^2 a unique quadric Q_{n-1}^2 through C^{2n}, whose section it is; and the locus of the quadrics thus determined by any one Q_{n-2}^2 is an n-fold, S_n^4, which is clearly of order 4, and a quadric section of Γ_{n+1}^2, since it does not contain Ω_{n-1}, but meets the latter in Q_{n-2}^2. We thus obtain on Γ_{n+1}^2 a linear system $|S_n^4|$ of freedom $\frac{1}{2}n(n-3)$, the characteristic system of a linear system of freedom $\frac{1}{2}(n-1)(n-2)$, of which Γ_{n+1}^2 itself is one. It is clear that all these quadrics pass through F^{4n}, and that they are all the quadrics that do so.

An immediate corollary is:

THEOREM 1.6 *If all the hypothesis of* 1.5 *holds, and if in addition the general curve of the canonical pencil has neither a* g_3^1 *nor a* g_5^2, F^{4n} *is the complete intersection of the linear system of quadrics found in* 1.5.

(The notation g_n^r is used as usual for a linear series of order n and freedom r.)

For precisely in this case, the general C^{2n} is the complete intersection of the system of quadrics through it in its ambient.

It would be agreeable to be able to conclude at this point that under these circumstances the system of quadrics has a subsystem of freedom $\frac{1}{2}n(n-3)$ intersecting in a threefold, whose complete intersection with Γ_{n+1}^2 would then be F^{4n}, so that the threefold would be G_3^{2n} and the surface would be that referred to in Theorems 1.2 and 1.3. This however does not seem to be easy for $n > 4$; for $n = 4$ it is obvious, since in this case F^{16} is the complete intersection of a linear system of quadrics in [6], of freedom 3, whereas any system of quadrics in [6] of freedom 2 has an intersection which is at least threefold. For higher values of n it is not clear that $\frac{1}{2}(n-1)(n-2)$ quadrics need have an intersection which is as much as threefold. For $n = 5$ for instance, as we shall see later, if F^{20} is the complete intersection of a general G_3^{10} with Γ_6^2, G_3^{10} is in turn a complete quadric section of a certain quintic fourfold, U_4^5, which is the complete intersection of a linear system of quadrics, of freedom 4. Thus in the system of all quadrics through F^{20}, of freedom 6, no subsystem of freedom 5 can have a G_3^{10} as its intersection, nor consequently a threefold intersection at all, which does not contain this particular subsystem of freedom 4.

Some idea of the relation between the surfaces considered in Theorems 1.2 and 1.3 and a possible more general type of F^{4n} can be obtained from the following result:

THEOREM 1.7 *If as before* F^{4n} *is the bicanonical model of a surface of genera* $p = 2$, $p^{(1)} = n + 1 > 3$, *with irreducible canonical pencil, its projection* $'F^{4n}$ *into* [4] *from a general* $[n-3]$, O_{n-3}, *lying in* Ω_{n-1}, *is the complete section of* $'\Gamma_3^2$ (*the projection of* Γ_{n+1}^2) *by a hypersurface of order* $2n$. *The projections* $'A_1, \ldots, 'A_n$ *of* $A_1, \ldots, A_n$ *are pinch points of* $'F^{4n}$, *the principal tangent at each being the vertex line* $'\omega$ *of* $'\Gamma_3^2$; $'F^{4n}$ *has a double curve* $'d^{n(4n-7)}$, *of order* $n(4n-7)$, *which is the complete intersection of* $'F^{4n}$ *with a surface* $'\Sigma^{4n+7}$ *lying on* $'\Gamma_3^2$; $'\Sigma^{4n+7}$ *is in turn*

the residual section of $'\Gamma_3^2$ *by a hypersurface of order* $2n-3$ *through one of its generating planes. The necessary and sufficient condition for* F^{4n} *to have a double rational curve* K^n *lying in a generating* $[n]$ *of* Γ_{n+1}^2 *is that* $'\Sigma^{4n-7}$ *breaks up into a surface* $'\Sigma^{4n-8}$*, complete intersection of* $'\Gamma_3^2$ *with a hypersurface of order* $2n-4$*, together with a generating plane of* $'\Gamma_3^2$*, the part of the double curve* $'d^{n(4n-7)}$ *which is the intersection of* $'F^{4n}$ *with this plane being the projection of the double curve* K^n *of* F^{4n}*.*

It is obvious that the projection of Γ_{n+1}^2 from O_{n-3} is $'\Gamma_3^2$; since Ω_{n-1} passes through O_{n-3} it is projected into a line $'\omega$, and every generating $[n]$ of Γ_{n+1}^2 is projected into a plane through $'\omega$, and of these planes two lie in a general [3] through $'\omega$. Each curve C^{2n} of the canonical pencil is projected into a plane curve $'C^{2n}$ in the corresponding generating plane of $'\Gamma_3^2$, which touches $'\omega$ in each of the points $'A_1, \ldots, 'A_n$; these are accordingly pinch points, $'\omega$ being the principal tangent at each. Since the curves $'C^{2n}$ have no variable intersection with $'\omega$, their locus $'F^{4n}$ is the complete intersection of $'\Gamma_3^2$ with a hypersurface of order $2n$. $'C^{2n}$ being of genus $n+1$ has $2n(n-2)$ double points, which (since the linear sections are part of the canonical series) are its complete intersection with a curve $'s^{2n-4}$ of order $2n-4$. The locus of the ∞^1 curves $'s^{2n-4}$ is a surface $'\Sigma$ lying on $'\Gamma_3^2$, whose complete intersection with $'F^{4n}$ is the double curve of the latter, or part of it, any residual part of the double curve lying wholely in one or more generating planes of $'\Gamma_3^2$, and having no variable intersection with the general plane.

On the other hand the general hyperplane section of $'F^{4n}$ is a curve $'f^{4n}$, complete section by a surface of order $2n$, of the cone $'\Gamma_2^2$, hyperplane section of $'\Gamma_3^2$. Thus to be of genus $3n+1$ (as it is), $'f^{4n}$ must have $n(4n-7)$ double points, so that this is the order of the double curve on $'F^{4n}$. As $'f^{4n}$ is the complete intersection of two surfaces of orders 2, $2n$, its complete canonical series is traced residually by surfaces of order $2n-2$ through its double points; on the other hand as the canonical series on the corresponding hyperplane section of F^{4n} is traced by quadrics through a generator of Γ_{n+1}^2, the residual intersection of $'f^{4n}$ with a quadric through a generator of $'\Gamma_3^2$, i.e., its complete intersection with a twisted cubic on $'\Gamma_2^2$, is a canonical set. This clearly means that the double points of $'f^{4n}$ are its complete intersection with a curve $'\sigma^{4n-7}$ on $'\Gamma_3^2$, obviously unique, which together with a general twisted cubic makes up the section of $'\Gamma_2^2$ by a surface of order $2n-2$, and is thus itself the residual section by a surface of order $2n-3$ through one generator.

Now if k of the curves $'s^{2n-4}$ pass through a general point of $'\omega$, the surface $'\Sigma$ generated by them has $'\omega$ as k-ple line, and is of order $4n-8+k$; for even values of k it is the complete section of $'\Gamma_3^2$ by a hypersurface of order $2n-4+\frac{1}{2}k$, for odd values the residual section by one of order $2n-4+\frac{1}{2}(k+1)$ through a generating plane. It is clear however that the hyperplane section of $'\Sigma$ is the curve $'\sigma^{4n-7}$, or part of it, any residual part consisting of the sections of any generating planes of $'\Gamma_3^2$ that contain double curves of $'F^{4n}$. Comparing the orders of $'\sigma$, $'\Sigma$ we see that $k \leqslant 1$. If $k=1$, $'\Sigma$ is of order $4n-7$ and is the

residual section of $'\Gamma_3^2$ by a hypersurface of order $2n - 3$ through one generating plane; its complete intersection with $'F^{4n}$ is the whole double curve $'d^{n(4n-7)}$ of the latter, which passes simply through the pinch points $'A_1, \ldots, 'A_n$ and meets each generating plane of $'\Gamma_3^2$ residually in the $2n(n-2)$ double points of $'C^{2n}$. If on the other hand $k = 0$, $'\Sigma$ is of order $4n - 8$ and is the complete section of $'\Gamma_3^2$ by a hypersurface of order $2n - 4$; its intersection with $'F^{4n}$ is a double curve $'d^{4n(n-2)}$ on the latter, locus of the double points of $'C^{2n}$, which does not meet $'\omega$; there is thus a further double curve $'K^n$, which is the complete intersection of $'F^{4n}$ with a particular generating plane $'X_2$ of $'\Gamma_3^2$. In the latter case $'K^n$ doubled, and branching at those pinch points of $'F^{4n}$ which lie in it, is a curve of the pencil $|'C^{2n}|$, i.e., it is the projection of the canonical curve C^{2n} of genus $n+1$ in the generator X_n of Γ_{n+1}^2 corresponding to $'X_2$. Since it is birationally equivalent to a double curve however, this C^{2n} is hyperelliptic, and (being the canonical model) is itself a double rational curve K^n, the complete intersection of X_n with F^{4n}. Conversely, of course, if F^{4n} has a double curve K^n in a generator X_n of Γ_{n+1}^2, this projects into a constituent $'K^n$ of the double curve of $'F^{4n}$, which is the complete intersection of $'F^{4n}$ with $'X_2$, the residual constituent being the complete intersection of $'F^{4n}$ with a hypersurface of order $2n - 4$.

We may plausibly conjecture that the existence of the double curve K^n on F^{4n} is a sufficient (as it is clearly a necessary) condition for F^{4n} to be the complete intersection of Γ_{n+1}^2, G_3^{2n}. To prove this it would be necessary to show that the hypersurface of order $2n - 4$ whose intersection with $'\Gamma_3^2$ is $'\Sigma^{4n-8}$ can be chosen so as to have on it a surface $\Delta^{2n(n-2)}$, whose complete intersection with $'\Gamma_3^2$ is $'d^{4n(n-2)}$, and further that the hypersurface $'G_3^{2n}$ whose intersection with $'\Gamma_3^2$ is $'F^{4n}$ can be so chosen as to have $\Delta^{2n(n-2)}$ as double locus. $'G_3^{2n}$ would then be the projection of a G_3^{2n}, projective model of the linear system traced residually on $'G_3^{2n}$ by hypersurfaces of order $2n - 3$ through its double surface $\Delta^{2n(n-2)}$. These two results however do not seem easy to prove, and I have not succeeded in establishing the sufficiency of the existence of the double curve for F^{4n} to be the complete intersection of Γ_{n+1}^2, G_3^{2n}, except for $n < 5$ (in which case it turns out that the property that $A_1, \ldots, A_n$ are pinch points is itself a sufficient condition.) The methods of proof however are different for the different values of n, and must be postponed until we come to consider the various values of n separately.

Meanwhile however we may remark that a number of the surfaces of genus 2 with canonical pencil of irreducible hyperelliptic curves, which I have studied in a recent paper (**4**), come under the specification of the special surfaces of 1.2, 1.3. For in the first place, a double rational ruled surface R_2^n, branching along a curve β^{2n+4} of order $2n+4$ which meets each generator in four points, and which is consequently the residual section of R_2^n by a quartic hypersurface through $2n - 4$ generators, is a surface G_3^{2n}, since the generators and hyperplane sections of R_2^n form a base for curves on it, and each of these systems is clearly adjoint to itself on the double surface. Consequently a double rational threefold R_3^n, branching along a surface B^{2n+4} which is its residual section by a quartic hyper-

surface through $2n - 4$ of its generating planes, is a G_3^{2n}. If now we consider a surface Φ^{2n}, the intersection of Γ_{n+1}^2 with R_3^n, and choose the branch surface B^{2n+4} so as to touch Φ^{2n} along the curve K^n traced on it by a particular generating $[n]$, X_n, of Γ_{n+1}^2 (e.g., but not necessarily, by letting it break up into the section of R_3^n by the tangent hyperplane Y_{n+1} to Γ_{n+1}^2 along X_n, together with the residual section of R_3^n by a cubic hypersurface through $2n - 4$ generating planes), then K^n is not a proper part of the branch curve of the double Φ^{2n}, since it counts twice in the intersection of B^{2n+4} with Φ^{2n}; and the branch curve of the double Φ^{2n} is its residual section by a cubic hypersurface through $2n - 4$ of the conics traced by the generating planes of R_3^n. Φ^{2n} has of course n double points $A_1, \ldots, A_n$, the intersections of R_3^n with the vertex Ω_{n-1} of Γ_{n+1}^2; and these are isolated branch points on the double Φ^{2n}, since a general curve on Φ^{2n} which passes simply through one of them does not touch B^{2n+4} there but intersects it simply, and therefore has a branch point there, regarded as a curve on the double surface. This double Φ^{2n} is thus precisely what I called the standard case of the bicanonical surface of genera $p = 2$, $p^{(1)} = n + 1$, with hyperelliptic canonical pencil, in the paper referred to.

Again, as the double plane with general sextic branch curve is the general G_2^2, the double Veronese surface whose branch curve is a general cubic section is one type of G_2^8, and one type of G_3^8 will be the double cone V_3^4 (projecting the Veronese surface from a point in [6]) whose branch surface is a general cubic section and having also an isolated branch point at its vertex (this last is necessary in order to ensure that every curve on V_3^4 shall have an even number of branch points, as it must for the double locus to exist at all). If now Φ^8 is the complete intersection of V_3^4, Γ_5^2, and the branch surface is again made to touch Φ^8 along K^4 (e.g., but not necessarily, by breaking up into the section of V_3^4 by Y_5 together with a general quadric section) the double Φ^8 has as branch curve a general complete quadric section, together with isolated branch points at its four double points, and is precisely what I called exceptional case no. xviii in that paper.

The seven other exceptional cases enumerated in the paper for $p = 2$ clearly do not give surfaces Φ^{2n} which are the complete intersection of Γ_{n+1}^2 with a threefold which, doubled and suitably branching, could be regarded as a G_3^{2n}; since any such threefold must have rational $[n]$ sections, and it is familiar that the only n-ic threefolds in $[n + 2]$ having this property are R_3^n and (for $n = 4$) V_3^4. It will remain to be considered, for any of these other exceptional cases, whether there can be a G_3^{2n} whose intersection with Γ_{n+1}^2 consists of the Φ^{2n} in question, counted twice, and if so what ought to be regarded as the branch curve when this situation is arrived at as the limit of a variable simple intersection of order $4n$.

2. The cases $n = 1$ ($p^{(1)} = 2$) and $n = 2$ ($p^{(1)} = 3$). These cases having been studied by Enriques (**7**, pp. 304, 312), comparatively little remains to be said of them; but a few remarks are worth making. There is one type of surface for $n = 1$, namely the double quadric cone Γ_2^2 in [3], branching along a general

quintic section, and having an isolated branch point at the vertex (**4**, p. 208), and this can be thought of as the intersection of Γ_2^2 with the G_3^2 consisting of the ambient [3] doubled and having as branch surface a sextic surface which touches Γ_2^2 along a generator. For $n = 2$ there are two types of F^8; one is the double Φ^4, intersection of Γ_3^2 with another quadric in [4] (having two double points, and a pencil of conics passing through both of them, traced by the generating planes of Γ_3^2) whose branch curve is a general cubic section, and having also isolated branch points at the two double points; and this is precisely our standard case (**4**, p. 207) of the surface with hyperelliptic canonical pencil. The other is the section of Γ_3^2 by a quartic hypersurface which touches it along a conic in a generating plane, which is of course the surface F^8 given by 1.2. It is worth remarking that 1.7 shows clearly why, in this particular case, there can be no more general F^8, not a complete section of Γ_3^2; for on the one hand as the surface is already in [4] no projection is involved, and F^8, $'F^8$ of 1.7 are the same surface, so that as $'F^{4n}$ is in any case a complete section of $'\Gamma_3^2$, F^8 is a complete section of Γ_3^2; on the other hand the surface $'\Sigma$, of order $4n - 7 = 1$, lying on Γ_3^2, whose intersection with F^8 is the whole double curve of the latter, is just a generating plane of Γ_3^2, and there can be no question of its failing to break up into a plane and a residual surface, complete section of Γ_3^2 by a hypersurface of order $2n - 4 = 0$, since in this case this residual surface is null.

It may also be pointed out here, with all diffidence, that Enriques appears to be wrong when he says (**7**, p. 315) that these two types of surface form distinct families with the same number of moduli; in fact,

THEOREM 2.1 *The regular surfaces of genera $p = 2$, $p^{(1)} = 3$, whose canonical pencil is irreducible and hyperelliptic, are a subfamily of those whose canonical pencil consists of general irreducible curves of genus* 3.

In other words, the general double Φ^4, with branch points at its two double points and branch curve which is a general cubic section, is contained in the family of sections of Γ_3^2 by a quartic touching it along a conic in a generating plane, and can be obtained as the limit of a variable surface of this latter type.

For let $(x_0, \ldots, x_4)$ be a homogeneous coordinate system in [4], so chosen that the equation of Γ_3^2 is

$$x_0 x_2 = x_1^2,$$

and let $\phi_2 = 0$ be any quadric whose intersection with Γ_3^2 is Φ^4, and $f_3 = 0$ any cubic whose intersection with Φ^4 is the branch curve of the double surface. This double Φ^4 can be taken as the section by Γ_3^2 of the double quadric $\phi_2 = 0$, with branch surface consisting of its sections by the hyperplane $x_0 = 0$ and the cubic $f_3 = 0$; since as the former partial branch surface touches Φ^4 along the conic K^2 traced by the plane $x_0 = x_1 = 0$, this conic will contribute nothing to the branching of the double Φ^4 except isolated branch points at the two nodes (as in the example considered at the end of §1.) But the double quadric so branching is the limit for $\lambda \to 0$ of the variable quartic

$$\phi_2^2 + \lambda x_0 f_3 = 0,$$

which for a general value of λ is irreducible and simple, and touches $x_0 = 0$, and hence Γ_3^2, along K^2. Thus this pencil of quartics traces on Γ_3^2 a pencil of surfaces F^8 of genera $p = 2$, $p^{(1)} = 3$, whose general member is non-singular except for the double conic K^2, whereas one surface of the pencil is the double Φ^4 from which we started, which was the most general of its kind. It is clear therefore that the whole family of double Φ^4's is contained as a subfamily in that of the F^8's.

Enriques' arguments to the contrary are twofold. He projects F^8 into an octavic surface $'F^8$ in [3], with two coincident and coplanar fourfold lines, and a double conic, and then says that the projection of Φ^8 into [3] is a quartic surface with a double conic, which cannot (counted twice) be the limit of a variable $'F^8$ unless the double conic reduces to a tacnodal line; but he seems to have forgotten that the projection of F^8 into $'F^8$ was made, not from a general point of [4], but from a general point of Γ_3^2, and that when Φ^4 is similarly projected from a point of Γ_3^2, the double conic of the projected surface does in fact reduce to a tacnodal line. Secondly, Enriques says that each of these families of surfaces has 24 moduli, without stating in either case how this figure is arrived at. For the double Φ^4, I think it is correct, as there are ∞^1 projectively distinct Φ^4's, as can be seen from the plane mapping by cubics with five base points $X_1, \ldots, X_5$, of which X_2 is in the neighbourhood of X_1, and the other three are in a line; the whole figure is projectively determined by the cross ratio of the lines joining X_1 to the other four points; Φ^4 has ∞^{24} cubic sections, of which however only ∞^{23} are projectively distinct, as the surface has ∞^1 projective transformations into itself (in the plane mapping, the pencil of homologies with centre X_1 and axis $X_3 X_4 X_5$); there are thus ∞^{24} projectively distinct figures consisting of Φ^4 together with a cubic section of itself. On the other hand it seems to me that F^8 has 26 moduli. In the first place there are ∞^6 lines Ω_1, each of which is the vertex of ∞^5 cones Γ_3^2, each of which has ∞^1 generating planes X_2, in each of which are ∞^5 conics K^2. Thus the cone Γ_3^2 and the conic K^2 can be chosen in ∞^{17} ways. The quartic hypersurfaces touching Γ_3^2 along K^2 are ∞^{48}, since every such quartic must trace on Y_3 (the tangent hyperplane to Γ_3^2 over X_2) a quartic surface with K^2 as double conic, of which there are ∞^{13}, as they are birationally equivalent to the quadric sections of a quadric in [4]; while there are ∞^{35} quartics in [4] tracing any given quartic surface on Y_3. On the other hand the quartic hypersurfaces tracing any given F^8 on Γ_3^2 are ∞^{15}; there are thus $\infty^{17+48-15} = \infty^{50}$, or ∞^{26} projectively distinct, surfaces F^8, since there are ∞^{24} projective transformations in [4].

3. The case $n = 3$ $(p^{(1)} = 4)$. For $n > 2$ we have as yet no absolute guarantee that there are any surfaces of the required genera, other than those whose canonical curves are hyperelliptic, though of course the presumption is that there are; the construction of 1.2 requires contact of G_3^{2n} with Γ_{n+1}^2 of a kind whose possibility is not obvious, and we have still no information as to

whether the more general type of surface envisaged in 1.7 can exist at all. We shall therefore begin by actually constructing some surfaces, of both kinds, for $n = 3$.

THEOREM 3.1 *A G_3^6 can be constructed in* [5] *to touch a Γ_4^2 along a rational cubic curve K^3 lying in a generating* [3] *X_3 of Γ_4^2, so that their intersection F^{12} is the bicanonical model of a regular surface of genera $p = 2$, $p^{(1)} = 4$.*

Since every non-hyperelliptic canonical curve of genus 4 is the complete intersection of a quadric and a cubic surface in [3], the general G_2^6 and G_3^6 are the complete intersections of a quadric and a cubic hypersurface in [4], [5] respectively. Since Γ_4^2 has the same tangent hyperplane Y_4 at all points of X_3, for G_3^6 to touch Γ_4^2 along K^3 is the same thing as for it to touch Y_4 along K^3, i.e., for its section by Y_4 to be a surface, virtually G_2^6, that is the intersection of a quadric and a cubic, but having K^3 as double curve, so that its hyperplane sections are not of genus 4 but elliptic, and the surface must be the projection of the sextic del Pezzo surface from some line. (We recall for comparison that in the case $n = 2$ the corresponding surface, section of G_3^4 by Y_3, being a quartic with the double conic K^2, was a projection of the quartic del Pezzo surface.)

We first show therefore that the sextic del Pezzo surface U_2^6 can be projected from a suitably chosen line l to give a surface $'U_2^6$ in [4] which has a double rational cubic curve K^3, forming a complete hyperplane section, and that $'U_3^6$ is the complete intersection of a quadric with a cubic hypersurface.

In the first place, just as the normal elliptic quartic E^4 has four points (the vertices of the four cones in the pencil of quadrics whose intersection it is) from each of which it projects into a double conic, the normal elliptic $2n$-ic curve E^{2n} has n^2 $[n-2]$'s from each of which it projects into a double K^n, image of one of the n^2 quadratic involutions on E^{2n} which have the property that any n pairs of the involution are together a hyperplane section of the curve.

Now let E^6 be a general hyperplane section of U_2^6, and let l be any one of the nine lines from which it projects into a double cubic K^3, image of an involution I^2; the projection of U_2^6 from l is a sextic surface $'U_2^6$ in [4], with the double curve K^3 constituting its whole section by a hyperplane X_3. We shall show that $'U_2^6$ is in fact the intersection of a quadric and a cubic. For U_2^6 has on it two homaloidal nets of rational cubics (represented, when U_2^6 is mapped on a plane by cubics with three base points, by the lines of the plane and the conics through the base points); the cubics of either net that join pairs of I^2 are a pencil with a base point on E^6, say P′, P″ for the two nets; and since the two nets are residual with respect to hyperplane sections of U_2^6, P′ P″ is also a pair of I^2. These two pencils of cubics appear on $'U_2^6$ as pencils of *plane* cubics, with double points on K^3, and base points at a point P of K^3, projection of P′ and P″. Every plane containing a cubic of one system meets every plane containing a cubic of the other in a line (which of course passes through P), since two cubics on U_2^6 one of each net, have two intersections. Thus the two systems of planes containing the plane cubics on $'U_2^6$ are the two systems of generating planes of a

quadric cone R_3^2, with point vertex P. $'U_2^6$ lies on this quadric, and since it traces a cubic curve on every generating plane, is its complete section by a cubic hypersurface Θ_3^3. (This is of course not determinate, but can be taken to be a general member of the linear system of freedom 5 which all trace the same surface on R_3^2.)

We can now take X_3 to be a generating [3] of Γ_4^2 in [5], and the ambient [4] of $'U_2^6$ to be the tangent hyperplane Y_4 to Γ_4^2 at all points of X_3. If now Q_4^2 is a general quadric whose section by Y_4 is R_3^2, and Θ_4^3 a general cubic whose section by Y_4 is Θ_3^3, the intersection of Q_4^2, Θ_4^3 is a G_3^6, whose section by Y_4 is $'U_2^6$; thus G_3^6 touches Y_4, and hence Γ_4^2, along K^3, so that the surface of intersection F^{12} of G_3^6, Γ_4^2 is precisely as specified in 1.2. Theorem 3.1 is thus proved.

Before investigating what other surfaces may exist for $n = 3$ it is convenient to recall briefly some properties of the threefold loci ${}^{\pi}H_3^m$, of order m, having hyperelliptic curve sections of genus π, and not generated by a hyperelliptic pencil of planes. These were considered in rather general terms by Enriques (**6**) long ago; proofs of any statement made here which may not appear obvious will be found in a recent paper of my own (**5**).

The surface ${}^{\pi}H_2^m$ of order m in $[m - \pi + 1]$ ($\pi > 2$, $\pi + 2 < m < 4\pi + 4$), with hyperelliptic hyperplane sections, and not ruled, was studied by Castelnuovo (**1**); it is rational, being mapped on a plane (in general) by $(\pi + 2)$-ic curves with one π-ple and $4\pi + 4 - m$ simple base points, and has a pencil of conics, represented by the lines through the π-ple base point, which trace the unique g_2^1 on each hyperplane section. Enriques showed that any threefold ${}^{\pi}H_3^m$ whose general hyperplane section is ${}^{\pi}H_2^m$, is rational and has on it a pencil of quadrics $|Q_2^2|$ which trace the unique pencil of conics on each hyperplane section. The ambient [3]'s of these generate a normal rational fourfold $R_4^{m-\pi-1}$, on which ${}^{\pi}H_3^m$ is coresidual to a quadric section, together with $2\pi + 2 - m$ of its generating [3]'s. The projection of ${}^{\pi}H_3^m$ from a general point of itself is a ${}^{\pi}H_3^{m-1}$; not, however, the general ${}^{\pi}H_3^{m-1}$, since one quadric surface of its pencil $|Q_2^2|$ breaks up into a pair of planes, arising respectively from the neighbourhood of the centre of projection and from the Q_2^2 through this point, whereas the general ${}^{\pi}H_3^{m-1}$ has no such reducible Q_2^2. Any base point of the pencil $|Q_2^2|$ is a $(\pi + 1)$-ple point on ${}^{\pi}H_3^m$.

We will now prove:

THEOREM 3.2 *In* [5], *let* Θ_4^3 *be a cubic hypersurface containing a plane* Ω_2, *and having three non-collinear double points* A_1, A_2, A_3 *in this plane. There are three* [3]'*s*, $X_3^{(i)}$ $(i = 1, 2, 3)$ *through* Ω_2 *whose residual intersections with* Θ_4^3 *are quadric surfaces* $Q_2^{2(i)}$, *tracing on* Ω_2 *the three pairs of sides of the triangle* $A_1 A_2 A_3$. *If* Γ_4^2 *is a quadric cone with vertex* Ω_2, *and having* $X_3^{(1)}, X_3^{(2)}, X_3^{(3)}$ *as generators, then the residual intersection* F^{12} *of* Θ_4^3, Γ_4^2, *and another cubic hypersurface through the three quadrics* $Q_2^{2(i)}$, *is the bicanonical model of a regular surface of genera* $p = 2$, $p^{(1)} = 4$; *the base points of the canonical pencil are* A_1, A_2, A_3, *which are simple points on the surface,* Ω_2 *being the tangent plane at each of them.*

In the first place the quadric surfaces traced residually by [3]'s through a fixed plane on a cubic hypersurface through the plane trace on the plane a net of conics, in projective correspondence with the net of [3]'s through the plane. The necessary and sufficient condition for this net to have a base point at a point A of the plane is that A is a double point of the cubic; thus in the case of the cubic Θ_4^3 the net is that of all conics through A_1, A_2, A_3, three members of which are the pairs of sides of the triangle $A_1A_2A_3$, so that there are, as stated, [3]'s $X_3^{(1)}, X_3^{(2)}, X_3^{(3)}$, whose quadric residual sections $Q_2^{2(1)}, Q_2^{2(2)}, Q_2^{2(3)}$, trace these three degenerate concics on Ω_2, and thus touch Ω_2 in A_1, A_2, A_3 respectively. (Any [3] through Ω_2 cuts the quadric cone tangent to Θ_4^3 at A_i in a pair of planes, namely Ω_2 and the tangent plane to the quadric residual section; thus $X_3^{(i)}$ is the unique [3] whose section with the tangent cone at A_i consists of Ω_2 counted twice, namely the base of the pencil of hyperplanes which touch this cone along its pencil of generating lines in Ω_2.) The quadric residual sections $|Q_2^2|$ by the generating [3]'s of Γ_4^2 thus trace on Ω_2 a quadratic family of conics of which the three pairs of sides of the triangle are members, and whose envelope is accordingly a quartic with cusps at A_1, A_2, A_3. The intersection of Θ_4^3, Γ_4^2 is a special type of ${}^3H_3^6$, on which Ω_2 is double (generated by the quadratic family of conics, and hence having the three cusped quartic as locus of pinch points); for whereas the general hyperelliptic sextic curve of genus 3 in [3] is the residual section of a quadric surface by a quartic through two generators of one system, when the quadric is a cone this curve reduces to the complete section by a cubic through the vertex; so that though the general ${}^3H_3^6$ is the residual section of the general R_4^2, a cone with line vertex and two systems of generating [3]'s, by a quartic through two [3]'s of the same system, when R_4^2 becomes Γ_4^2, ${}^3H_3^6$ becomes its section by a cubic through Ω_2. The points A_i are quadruple points of ${}^3H_3^6$, being double on each of the intersecting hypersurfaces; the tangent cone to ${}^3H_3^6$ at A_i is in fact a (non-normal) R_3^4, generated by the tangent planes to the pencil of quadrics $|Q_2^2|$ at A_i; each of these planes meets Ω_2 in a line (the tangent to the conic traced by Q_2^2 on Ω_2), and Ω_2 is itself one of the family, being the tangent plane at A_i to $Q_2^{(i)}$; thus the cone is that projecting a normal rational R_2^4 in [5], with directrix line, from a point coplanar with the directrix line and a generator.

Now consider the surface F^{12}, residual section of ${}^3H_3^6$ by a general cubic hypersurface through $Q_2^{2(1)}, Q_2^{2(2)}, Q_2^{2(3)}$. F^{12} has no curve of intersection with Ω_2, since the three quadric surfaces meet this plane altogether in the lines A_2A_3, A_3A_1, A_1A_2, each twice, which accounts for the whole intersection of the secant cubic with the double Ω_2. Moreover, since the secant cubic meets Ω_2 in these three lines, it touches the plane in A_1, A_2, A_3; it is clear in fact that F^{12} touches Ω_2 in these three points; for the tangent planes to the three quadric surfaces at A_i are Ω_2 (tangent to $Q_2^{2(i)}$) and two other generating planes of the tangent cone R_3^4; these are joined by a hyperplane, which is necessarily the tangent hyperplane to the secant cubic, and whose residual intersection with R_3^4 is Ω_2 counted a second time; Ω_2 is thus the tangent plane at A_i to the residual

intersection F^{12}. On each quadric of the pencil $|Q_2^2|$ on ${}^3H_3^6$, F^{12} traces a sextic curve C^6 of genus 4, complete section of Q_2^2 by the secant cubic, since $Q_2^{2(1)}$ has no curve of intersection with the general Q_2^2; and this C^6 clearly touches Ω_2 in A_1, A_2, A_3, both because the secant cubic does so, and because F^{12} does so. The hyperplane sections $|f^{12}|$ of F^{12} are the double of the pencil $|C^6|$, i.e., any two curves of $|C^6|$ are a hyperplane section of F^{12}, and every hyperplane through Ω_2 meets F^{12} in two curves of $|C^6|$, since it meets Γ_4^2 in two generating [3]'s, and ${}^3H_3^6$ in two surfaces of the pencil $|Q_2^2|$.

We shall now show that the canonical series on the general f^{12} is traced on it residually by quadrics in its ambient [4], through its intersections with any one curve of the pencil $|C^6|$, i.e., with any generating [3] of Γ_4^2. For this purpose we consider the corresponding hyperplane section ${}^3H_2^6$ of ${}^3H_3^6$. The general ${}^3H_2^6$ is mapped on a plane by quintics with a triple base point X and ten simple base points $Y_1, \ldots, Y_{10}$, the lines through X representing the conics on the surface; since in the present case ${}^3H_2^6$ has a double line (the section of Ω_2) which with any two conics forms a hyperplane section, this is represented by a cubic curve on which all the base points lie. f^{12}, being the residual section of the surface by a cubic through three particular conics of the pencil, is mapped by a curve of order 12 with a sextuple point at X and triple points at $Y_1, \ldots, Y_{10}$; and on this the canonical series is traced by curves of order 9 with a quintuple base point at X and double base points at $Y_1, \ldots, Y_{10}$, which are just what represent the residual scetions of ${}^3H_2^6$ by quadrics through any one of its conics.

Thus the adjoint system to the hyperplane sections $|f^{12}|$ of F^{12} is traced on F^{12} residually by quadrics through any one Q_2^2 of the pencil on ${}^3H_3^6$, i.e., through any one C^6 of the pencil on F^{12}, and is accordingly the system $|2f^{12} - C^6| = |f^{12} + C^6|$, which means that $|C^6|$ is the canonical system on F^{12}. Theorem 3.2 is thus established.

THEOREM 3.3 *There is a type of* ${}^2H_3^5$ *in* [5], *residual intersection of* Γ_4^2 *with a cubic hypersurface through one of its generating* [3]'*s, with the following features: the pencil of quadrics* $|Q_2^2|$ *on* ${}^2H_3^5$ *trace on* Ω_2 *a pencil of conics with three base points* A_1, A_2, A_3, *and with a fixed tangent* k *in* A_3; *and one quadric of the pencil* $|Q_2^2|$ *breaks up into a pair of planes* κ, λ, *meeting* Ω_2 *in the lines* k, A_1A_2 *respectively. There are also cubic hypersurfaces* Θ_4^3, *containing the plane* λ, *containing also that quadric* Q_2^2 *of the pencil which traces on* Ω_2 *the pair of lines* A_2A_3, A_3A_1, *and further touching* ${}^2H_3^5$ *along a line* s *lying in the plane* κ *and passing through* A_3. *If* ${}^2H_3^5$, Θ_4^3 *satisfy these conditions, their residual intersection* F^{12} *is the bicanonical model of a surface of genera* $p = 2$, $p^{(1)} = 4$, *on which the canonical pencil is traced by the pencil* $|Q_2^2|$, *and has the base points* A_1, A_2, A_3, *of which the two former are simple points on the surface whereas* A_3 *is a pinch point.*

In the first place we will satisfy ourselves that a ${}^2H_3^5$ exists with the desired peculiarities. The residual section of Γ_4^2 by a general cubic through one generat-

ing [3] is a ${}^2H_3^5$, its pencil of quadric surfaces $|Q_2^2|$ being the sections of the cubic by the generating [3]'s of Γ_4^2, residual to Ω_2, and these trace on Ω_2 a pencil of conics, since Ω_2 is simple on ${}^2H_3^5$, and hence one Q_2^2 passes through a general point of it. The base points of this pencil of conics are double on the secant cubic and triple on ${}^2H_3^5$, the tangent cone at each being an R_3^3 (cone projecting a ruled cubic surface from a point), intersection of Γ_4^2 with the tangent cone to the secant cubic, residual to the common [3]; it is generated by the tangent planes to the quadrics $|Q_2^2|$, and has Ω_2 as directrix plane, on which the generating planes trace a pencil of lines.

Now let R_4^3 in [6] be a cone with line vertex l, and a directrix [3] on which its generating [3]'s trace the pencil of planes through l; the section of this by a general quadric is a ${}^2H_3^6$, with two triple points A_1, A_2, the intersections of l with the secant quadric; these are base points of the pencil $|Q_2^2|$ on ${}^2H_3^6$; there is also a quadric surface $\bar{Q}_2^2$ traced by the secant quadric on the directrix [3], and on this the pencil $|Q_2^2|$ trace the pencil of plane sections with base points A_1, A_2. If we now project ${}^2H_3^6$ into [5] from a point K of $\bar{Q}_2^2$, $\bar{Q}_2^2$ is projected into a plane Ω_2, R_4^3 into Γ_4^2 with Ω_2 as vertex, and of course ${}^2H_3^6$ into a ${}^2H_3^5$ lying on Γ_4^2. The conics traced by the projected pencil $|Q_2^2|$ on Ω_2 are the projections of those traced by the original pencil $|Q_2^2|$ on $\bar{Q}_2^2$, so that they form a pencil on conics in Ω_2, whose base points are the projections of A_1, A_2, together with the points A_3, A_4 arising from the two generators of $\bar{Q}_2^2$ through K. The neighbourhood of K on ${}^2H_3^6$ gives rise to a plane κ on ${}^2H_3^5$, passing through A_3, A_4, and the quadric Q_2^2 through K to a plane λ passing through A_1, A_2. We can thus get a ${}^2H_3^5$ answering our requirements by letting the secant quadric in [6] either touch the directrix [3] of R_4^3, so that $\bar{Q}_2^2$ is a cone and A_3, A_4 coincide, or touch l so that A_1, A_2 coincide. The notation we have used supposes the former, but the ${}^2H_3^5$'s obtained by these two specializations of the construction are in fact identical.

We remark also that at the point A_3, where the conics in Ω_2 have the common tangent k, the (cubic) tangent cone to ${}^2H_3^5$ has not a point vertex but the line vertex k, and any three of its generating planes form a hyperplane section. Ω_2 is one of these, and κ is another. We may call this cone Γ_3^3.

Consider now an arbitrary line s in κ, passing through A_3. Let X_3 be the generating [3] of Γ_4^2 which contains κ, λ, and Y_4 the tangent hyperplane to Γ_4^2 at all points of X_3. The tangent [3]'s to ${}^2H_3^5$ at points of s all contain κ, and all lie in Y_4, and thus form a pencil, each member of which touches ${}^2H_3^5$ in just one point, since the secant cubic has a double point in s, namely A_3, and thus no plane or [3] through s is bitangent to it. In this one-one (and hence projective) correspondence between the pencil of [3]'s through κ in Y_4 and their points of contact on s, the tangent [3] to Γ_3^3 at all points of κ corresponds to A_3, and X_3 to the point of intersection of s with λ, since X_3 clearly touches ${}^2H_3^5$ at all points of the line of intersection of κ with λ.

Now if a cubic hypersurface in [5] contains a line, its tangent hyperplanes at points of the line form a cone Γ'^2_4, since a general plane through the line touches

the cubic in two points; thus there is a unique plane through the line, the vertex of $\Gamma'_4{}^2$, which touches the cubic at all points of the line, i.e., whose intersection with the cubic consists of this line counted twice and some other line; and the [3]'s traced by the tangent hyperplanes in question on any one of them are just the pencil of [3]'s in this [4], passing through the vertex plane of $\Gamma'_4{}^2$, and in projective correspondence with their points of contact on the line. If therefore we make a cubic hypersurface $\Theta_4{}^3$ contain the plane λ and that particular quadric $Q'_2{}^2$ of the pencil $|Q_2{}^2|$ which traces the line pair A_3A_1, A_2A_3 on Ω_2, and make it also touch Y_4 in A_3 and κ at all points of s, so that its intersection with κ consists of s counted twice together with the line of intersection of κ with λ, the tangent hyperplanes to $\Theta_4{}^3$ at points of s will trace on Y_4 the pencil of [3]'s through κ, in projective correspondence with their points of contact on s; that at the point of intersection of s with λ traces X_3; so that it is only necessary to make $\Theta_4{}^3$ touch in two further points of s the tangent [3]'s to ${}^2H_3{}^5$, to ensure that it shall do so in all points of s, so that s is a double line on the intersection of $\Theta_4{}^3$ with ${}^2H_3{}^5$. All this imposes only 31 linear conditions, while the freedom of cubics in [5] is 55, so that there is an ample supply of cubics satisfying all the conditions.

The intersection of $\Theta_4{}^3$, ${}^2H_3{}^5$, residual to λ and $Q'_2{}^2$, is a surface F^{12} of order 12. At A_1, A_2 it has simple points with Ω_2 as tangent plane, since the tangent hyperplane to $\Theta_4{}^3$ at either of these contains two generating planes of the tangent cone to ${}^2H_3{}^5$ (namely λ and the tangent plane to $Q'_2{}^2$) and thus meets this cone further in its directrix plane Ω_2, which is accordingly the tangent plane to the residual intersection F^{12}. At A_3 on the other hand the tangent hyperplane to $\Theta_4{}^3$ is Y_4, which meets the tangent cone $\Gamma_3{}^3$ to ${}^2H_3{}^5$ in the planes κ (doubly) and Ω_2 (simply), the latter being the tangent plane to $Q'_2{}^2$. F^{12} has thus a double point at A_3 (as of course it must, since s is a double line) with κ counted twice as tangent cone; the natural assumption from this is that A_3 is a pinch point, which will become certain when it appears in the sequel that A_3 is an improper singularity. $\Theta_4{}^3$ traces on Ω_2 the three lines A_2A_3, A_3A_1, A_1A_2 (the first two being on $Q'_2{}^2$, the third on λ), and thus touches Ω_2 in A_1, A_2, A_3; it thus traces on the general surface $Q_2{}^2$ a sextic curve C^6 of genus 4, touching Ω_2 in A_1, A_2, A_3, and this curve belongs wholly to F^{12}, since neither λ nor $Q'_2{}^2$ has any curve of intersection with the general $Q_2{}^2$. On the other hand, λ and $Q'_2{}^2$ account for the whole intersection of $\Theta_4{}^3$ with Ω_2, so that F^{12} meets Ω_2 only in the three points A_1, A_2, A_3.

The general hyperplane section f^{12} of F^{12} can best be studied by means of the plane mapping of the corresponding hyperplane section ${}^2H_2{}^5$ of ${}^2H_3{}^5$. This is by quartics with a double base point X (lines through which represent the conics on the surface, sections of the pencil $|Q_2{}^2|$) and seven simple base points $Y_1, \ldots, Y_7$. The fact that the $R_3{}^2$ generated by the planes of the conics is not the general $R_3{}^2$ (cone with point vertex) but a $\Gamma_3{}^2$ with line vertex ω, section of Ω_2, means that there is a line ω on the surface, bisecant to the conics, which together with any two conics makes up a hyperplane section; and this in turn requires that

$Y_1, \ldots, Y_7$ all lie on a conic, the image of ω. Seven of the conics break up into line pairs, represented by XY_i and the neighbourhood of Y_i; we can suppose that the neighbourhood of Y_7 represents the section of the plane κ, and the line XY_7 that of λ. f^{12}, being the residual section of ${}^2H_2{}^5$ by a cubic through this last line and also through the conic, section of $Q'_2{}^2$, is mapped by a curve of order 10, with a quadruple point at X, triple points at $Y_1, \ldots, Y_6$, and a double point at Y_7; it has also a double point Z in the neighbourhood of Y_7, corresponding to the actual double point of f^{12} at its intersection with s. The canonical series is traced on this curve by septimics with a triple base point at X, double base points at $Y_1, \ldots, Y_6$, and simple base points at Y_7, Z, amongst which are those which have a double point at Y_7 and do not pass through Z; these last clearly represent residual sections of ${}^2H_2{}^5$ by quadrics through one of its pencil of conics; thus the canonical series on f^{12} is traced residually by quadrics through its intersections with any one surface $Q_2{}^2$, and the adjoint system to the hyperplane sections $|f^{12}|$ is traced residually on F^{12} by quadrics through any one curve C^6, i.e., it is the system $|2f^{12} - C^6| = |f^{12} + C^6|$ which means that the pencil $|C^6|$ is the canonical system. Theorem 3.3 is thus established; it remains only to note that the general f^{12} is of genus 10, which is what the genus of $|2C^6|$ would be if the three base points of $|C^6|$ were all simple points of the surface; A_3 is thus an improper singularity, and as it cannot be the intersection of two simple sheets (since the pencil $|C^6|$ which passes simply through A_3, and of which any two curves are a hyperplane section, is irreducible) it can only be a pinch point.

Before showing that the surfaces constructed in these three theorems include the bicanonical models of all surfaces of genera $p = 2$, $p^{(1)} = 4$, with irreducible and non-hyperelliptic canonical pencil, it is convenient to digress, and devote some study to a surface which exhibits many of the features which we should expect of our bicanonical surface F^{12}, but which proves nevertheless to be of genus 1, and not 2.

THEOREM 3.4 *There exist surfaces* $\bar{F}^{12}$ *in* [5], *lying on* $\Gamma_4{}^2$, *and having a pencil* $|C^6|$ *of canonical curves of genus* 4 *traced by the generating* [3]'*s of* $\Gamma_4{}^2$, *with three base points* A_1, A_2, A_3, *simple points of the surface, in each of which the tangent plane to the surface is the vertex* Ω_2 *of* $\Gamma_4{}^2$, *so that the base points are a semicanonical set on each curve of the pencil; but on which nevertheless the canonical system is not the pencil* $|C^6|$, *but contains only one curve, which breaks up into three conics, each forming part of a different curve of the pencil* $|C^6|$ *and each passing through one of the base points* A_1, A_2, A_3; *these three conics are exceptional curves, and the surface belongs tothe familiar series of surfaces with all genera equal to unity, the unique curve of whose reduced canonical system is the null curve, so that every linear system on the surface is adjoint to itself* (**7**, p. 247).

First we must show that there exists a type of ${}^2H_3{}^5$, special in two respects: the $R_2{}^4$ generated by the ambient [3]'s of its $Q_2{}^2$'s is $\Gamma_4{}^2$ (with plane instead of line vertex), and three of these $Q_2{}^2$'s break up into pairs of planes. The latter peculiarity is evidently ensured by obtaining ${}^2H_3{}^5$ as the projection of ${}^2H_3{}^8$ in

[8] (residual section of R_4^5 by a quadric through two of its generating [3]'s from three points K_1, K_2, K_3 of itself); the former can be ensured by choosing K_1, K_2, K_3 suitably. For the general R_4^5 is generated by the [3]'s joining corresponding points of three lines and a conic, projectively related; it thus has on it an R_3^3, generated by the planes joining corresponding points of the three lines, which is its residual section by a hyperplane through two generating [3]'s, and which of course meets each generating [3] in a plane. The quadric which cuts R_4^5 in two [3]'s and ${}^2H_3^8$, cuts R_3^3 in two planes and a ruled quartic surface R_2^4, which is accordingly the intersection of ${}^2H_3^8$ and R_3^3 on R_4^5. The conics on R_2^4 are traced by the planes of R_3^3, and are the sections by these planes of the Q_2^2's in the corresponding [3]'s of R_4^5. If now K_1, K_2, K_3 are not general points of ${}^2H_3^8$, but are on R_2^4, it is clear that the directrix lines a_1, a_2, a_3 of R_3^3 through K_1, K_2, K_3 respectively are projected into points A_1, A_2, A_3, that R_2^4, as well as every generating plane of R_3^3, is projected into the plane $\Omega_2 = A_1 A_2 A_3$, and that the generating [3]'s of R_4^5 are projected into [3]'s which all pass through Ω_2, so that the projection of R_4^5 is not the general R_4^2 with line vertex, but a Γ_4^2 with vertex Ω_2. The conics on R_2^4 are projected into a pencil of conics in Ω_2, which are the traces on it of the pencil of quadrics $|Q_2^2|$ on ${}^2H_3^5$, the projections of those on ${}^2H_3^8$; this pencil of conics has as base points A_1, A_2, A_3, and a fourth point B, the projection of the unique rational cubic curve b on R_2^4 through A_1, A_2, A_3. These four points are triple points of ${}^2H_3^5$, the tangent cone at each being generated by the tangent planes to the quadric surfaces $|Q_2^2|$, and having Ω_2 as directrix plane. The three plane pairs in the pencil $|Q_2^2|$ evidently trace on Ω_2 the three line pairs in the pencil of conics; if we denote by κ_i, λ_i respectively the planes arising from the neighbourhood of K_i and from the Q_2^2 through K_i on ${}^2H_3^8$, we see that as a_i and b both pass through K_i, A_i and B both lie in κ_i, while as the Q_2^2 through K_i meets a_j, a_k, in general points, λ_i passes through A_j, A_k. Thus κ_i, λ_i trace on Ω_2 the lines $A_i B$, $A_j A_k$ respectively.

Now consider the surface $\bar{F}^{12}$, residual section of this ${}^2H_3^5$ by a general cubic hypersurface through the planes $\lambda_1, \lambda_2, \lambda_3$. It is of order 12, and meets Ω_2 in no curve, since the whole intersection of the secant cubic with Ω_2 consists of the three lines $A_2 A_3, A_3 A_1, A_1 A_2$, traced by $\lambda_1, \lambda_2, \lambda_3$; $\bar{F}^{12}$ passes simply through A_1, A_2, A_3, its tangent plane in each of these being Ω_2, since the tangent hyperplane to the secant cubic at A_i contains the planes λ_j, λ_k, which are generating planes of the tangent cone to ${}^2H_3^5$, and thus cuts this cone residually in Ω_2, which is accordingly the tangent plane there to the residual intersection $\bar{F}^{12}$. Since $\lambda_1, \lambda_2, \lambda_3$ trace no curve on the general Q_2^2, $\bar{F}^{12}$ traces on each Q_2^2 its complete section by the secant cubic, a canonical curve C^6 of genus 4, touching Ω_2 in A_1, A_2, A_3. $\bar{F}^{12}$ traces on each plane κ_i a conic s_i, since κ_i meets the secant cubic in a cubic curve of which the intersection of κ_i, λ_i is part. The curve t_i traced by $\bar{F}^{12}$ on λ_i is accordingly a quartic, since s_i, t_i together form a curve of the pencil $|C^6|$ on $\bar{F}^{12}$; and it is clear that s_i touches Ω_2 in A_i, and t_i touches it in A_j, A_k.

All the properties of $\bar{F}^{12}$ so far found are consistent with (and indeed strongly suggest) the idea that it is the bicanonical model of a surface of genera $p = 2$, $p^{(1)} = 4$, $|C^6|$ being its canonical pencil. We shall show however that $\bar{F}^{12}$ is on the contrary of genus 1, its unique canonical curve consisting of the three conics s_1, s_2, s_3.

To prove this we shall consider as before the general hyperplane section $\bar{f}^{12}$ of $\bar{F}^{12}$ as a curve on the corresponding section ${}^2H_2{}^5$ of ${}^2H_3{}^5$. ${}^2H_2{}^5$ as we have seen is rational, being mapped on a plane by quartics with a double base point X and seven simple base points $Y_1, \ldots, Y_7$, the latter all lying on a conic. Seven conics of the pencil break up into pairs of lines; we may suppose that amongst these, the lines represented by the neighbourhoods of Y_5, Y_6, Y_7 are the sections of $\kappa_1, \kappa_2, \kappa_3$, and those represented by XY_5, XY_6, XY_7 are the sections of $\lambda_1, \lambda_2, \lambda_3$. Thus $\bar{f}^{12}$, being the residual section of ${}^2H_2{}^5$ by a cubic through these last three lines, is represented by a curve of order 9 with triple points at X, $Y_1, \ldots, Y_4$, and double at Y_5, Y_6, Y_7; on this curve the canonical series is traced residually by sextics with double base points at X, $Y_1, \ldots, Y_4$, and simple at Y_5, Y_6, Y_7; amongst which are those that break up into the conic through $Y_1, \ldots, Y_7$ (which has no residual intersection with the curve) and quartics with a double base point at X and simple base points at $Y_1, \ldots, Y_4$; and in this latter system, those which pass also through Y_5, Y_6, Y_7 trace on the curve sets corresponding to hyperplane sections of $\bar{f}^{12}$, together of course with the pairs of points coinciding in Y_5, Y_6, Y_7, which represent the pairs of points traced on $\bar{f}^{12}$ by s_1, s_2, s_3. Thus canonical sets are traced on $\bar{f}^{12}$ by all the reducible curves on $\bar{F}^{12}$ consisting of another hyperplane section together with the three conics s_1, s_2, s_3; which means that these three conics are together a curve of the canonical system. That there is no other curve of this system is fairly obvious, and becomes certain when we remark that the three conics are of negative grade, and are in fact exceptional curves. That the grade of s_i is -1 follows from the fact that as s_i, t_i are together a curve of the pencil $|C^6|$, the virtual intersections of s_i with itself are equivalent to its intersections with a general C^6, minus its intersections with t_i, which are two in number, namely its intersections with the line traced by λ_i on its ambient plane κ_i. Theorem 3.4 is thus established.

It is now comparatively easy to prove:

THEOREM 3.5 *Every regular surface on which is a pencil $|C|$ of irreducible non-hyperelliptic curves of genus* 4, *with three base points which are a semicanonical set on the general curve of the pencil, has as projective model of the system $|2C|$ one of the four surfaces constructed in Theorems* 3.1, 3.2, 3.3, 3.4.

In the first place, as in Theorem 1.1, the projective model of $|2C|$ is a surface F^{12} lying on $\Gamma_4{}^2$ in [5]; the curves $|C|$ are canonical sextics C^6 on this model, and are traced by the generating [3]'s of $\Gamma_4{}^2$, whose vertex Ω_2 they all touch in the base points A_1, A_2, A_3 of the pencil. Also, as in Theorem 1.4, these base points are simple points or pinch points of F^{12} according as the variable curve

of the pencil has variable or fixed tangent there. The general curve of $|2C|$ is of genus 10.

The general C^6 lies on a unique quadric surface Q_2^2 in its ambient [3], which traces on Ω_2 a conic passing through A_1, A_2, A_3 and touching there the tangents to C^6. If m of these conics pass through a general point of Ω_2, the threefold locus generated by the surfaces Q_2^2 is of order $4 + m$, having Ω_2 as m-ple locus, and is (for even values of m) the section of Γ_4^2 by a hypersurface of order $2 + \frac{1}{2}m$, passing $\frac{1}{2}m$-ply through Ω_2, or (for odd values) the residual section by a hypersurface of order $2 + \frac{1}{2}(m + 1)$ passing simply through a generating [3] and $\frac{1}{2}(m + 1)$-ply through Ω_2. We shall show that $m \leqslant 2$. For if the tangents at at least two of the base points to the curves $|C^6|$ and the conics in Ω_2 are fixed, the conics all coincide, and $m = 0$; if the tangents are variable at at least two base points, they correspond projectively, so that the conics are transformed by a quadratic transformation with base points at $A_1 A_2 A_3$ into a family of lines which trace projective ranges of points on at least two sides of the triangle; such a family of lines is either a pencil, or the tangents to a conic which touches the sides of the triangle, and the family of conics is thus either a pencil (giving $m = 1$) or a quadratic family enveloping a quartic with cusps at A_1, A_2, A_3 $(m = 2)$.

If $m = 0$, the locus of the quadrics Q_2^2 is a quadric section S_3^4 of Γ_4^2, and since F^{12} traces a C^6 of genus 4 on each Q_2^2, the virtual difference on S_3^4 between F^{12} and a cubic section must be of order zero, and have no intersection with the general Q_2^2, i.e., must be null; thus F^{12} is a cubic section of S_3^4, i.e., it is the complete intersection of Γ_4^2, another quadric, and a cubic, i.e., of Γ_4^2 with the G_3^6 which is the intersection of the second quadric with the cubic. Thus the surface is that constructed in Theorem 3.1.

In the case $m = 2$ on the other hand, when the conics in Ω_2 envelope a tricuspidal quartic, the locus of the quadrics Q_2^2 is the section of Γ_4^2 by a cubic through Ω_2, i.e., it is a ${}^3H_3^6$, and is precisely the ${}^3H_3^6$ considered in Theorem 3.2. On this the virtual difference between F^{12} and a cubic section must be of order 6, and must as before have no intersection with the general Q_2^2, i.e., F^{12} is the residual section of ${}^3H_3^6$ by a cubic through some surface of order 6, which consists of quadrics Q_2^2 or planes forming parts of these. Moreover, since each C^6 touches Ω_2 in A_1, A_2, A_3, the secant cubic must touch Ω_2 in these points, i.e., must cut it in the lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$, and its total intersection with the double Ω_2 on ${}^3H_3^6$ consists of these three lines each counted twice; and this must be accounted for entirely by the trace on Ω_2 of the sextic surface of intersection residual to F^{12}, since F^{12} has no curve of intersection with Ω_2. But the three pairs of these three lines are the traces on Ω_2 of particular quadrics of the pencil $|Q_2^2|$; these three quadrics thus form the residual intersection, i.e., F^{12} is the residual section of ${}^3H_3^6$ by a cubic through these three quadrics, and is thus the surface constructed in Theorem 3.2.

If on the other hand the conics traced by $|Q_2^2|$ on Ω_2 are a pencil, they have

either a fourth base point B, distinct from A_1, A_2, A_3, or a fixed tangent k in one of these, say A_3. In either case $m = 1$, and the locus of the quadrics $|Q_2^2|$ is a ${}^2H_3^5$, residual section of Γ_4^2 by a cubic through one generating [3]. F^{12} must differ from a cubic section by some residual surface of total order 3, which as before must consist entirely of quadrics Q_2^2 or planes forming part of these; as before, the secant cubic must cut Ω_2 in the three lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$, and these must be entirely accounted for as the trace on Ω_2 of the residual intersection of order 3.

If the conics in Ω_2 have a fixed tangent k in A_3, two of them are line pairs, namely $A_2 A_3$, $A_3 A_1$ and $A_1 A_2$, k; thus the Q_2^2 tracing the former line pair must be part of the residual surface, and that tracing the latter must break up into two planes κ, λ, tracing k, $A_1 A_2$ respectively, of which λ must be the other part of the residual surface. A general F^{12} however, residual to this quadric and plane, would trace a conic on κ, which (by an analysis of the properties of the hyperplane section of the surface, on that of ${}^2H_3^5$, precisely similar to that in Theorem 3.3, merely omitting the double point Z in the neighbourhood of Y_7) can be seen to be a fixed part of the canonical system, the variable part being the pencil $|C^6|$; thus though the surface is of genus 2, it is not the bicanonical model, nor is the canonical pencil irreducible; moreover, the general curve of $|f^{12}| = |2C^6|$ is of genus 11, instead of 10, showing that A_3 is a proper singularity. The only way to make this singularity improper, and the general hyperplane section of genus 10, is to give the surface a double line passing through A_3; and this double line must lie in κ, replacing the conic which would otherwise be traced on the surface by κ, since it is precisely the pair of points traced by this conic on the general hyperplane section which we need to subtract from the canonical series on the latter. Thus we have the surface constructed in Theorem 3.3.

Finally, if the pencil of conics in Ω_2 has a fourth base point distinct from A_1, A_2, A_3, the lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ form parts of three distinct conics of the pencil, so that the residual surface must consist of three planes, meeting Ω_2 in these three lines respectively, and forming parts of three distinct quadrics of the pencil $|Q_2^2|$. Thus we have the surface $\bar{F}^{12}$ of Theorem 3.4.

From this result an obvious corollary is

THEOREM 3.6 *Every regular surface of genera $p = 2$, $p^{(1)} = 4$, has as its bicanonical model one of the following four surfaces:*

I. *That constructed in Theorem* 3.2, *without singularities*;

II. *That constructed in Theorem* 3.3, *with a double line, passing through one of the base points of the canonical pencil, which is a pinch point, the other two being simple*;

III. *That constructed in Theorem* 3.1, *with a double rational cubic curve passing through all three base points of the canonical pencil, all of which are pinch points*;

IV. *That given in my paper referred to above* (**4**), *on which the general curve of the canonical pencil is hyperelliptic; the surface is the double ${}^2H_2^6$, complete intersection of Γ_4^2 with a general R_3^3, with branch curve of order* 14, *residual section of*

${}^2H_2^6$ by a cubic hypersurface through two of its pencil of conics, and having also isolated branch points at its three nodes, the intersections of Ω_2 with R_3^3, which are the base points of the canonical pencil.

The point of view adopted in Theorem 1.8 makes it natural to regard type I as the general case, of which all the others are specializations. Whether, however, every surface of types II, III, IV can be obtained as the limit of a variable surface of type I seems to be quite a difficult problem, to which we shall not attempt a definite answer. In this connexion, however, two simple and obvious remarks present themselves:

(i) Though type III seems to be in some sense more special than type II, the general surface of type III cannot be the limit of a variable surface of type II, as if this assumes such a limiting form the double cubic curve of the latter will break up into a line (limit of the double line on the variable surface) and a conic.

(ii) Similarly, though type IV is in fact, as was remarked at the close of §1, the complete intersection of Γ_4^2 with a G_3^6, in this case the double R_3^3, with branch surface of order 10 which is its residual section by a quartic hypersurface through two generating planes, the general surface of type IV is not the limit of a variable surface of type III, since the general R_3^3 is not, counted twice, the complete intersection of a quadric and a cubic. Every quadric hypersurface containing the general R_3^3 is in fact an R_4^2 with line vertex (which is any one of the ∞^2 directrix lines of R_3^3); and this has on it two distinct systems of R_3^3's, whose generating planes lie respectively in the two systems of generating [3]'s of R_4^2; and a cubic hypersurface through an R_3^3 of one system cuts R_4^2 residually in one of the other system. R_3^3 can only be the complete intersection of a quadric and a cubic if it is a cone with (in general) a point vertex and a directrix plane; the quadric is then a cone with plane vertex, which is the directrix plane of R_3^3.

References

1. G. Castelnuovo, *Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche*, Rend. Circ. Mat. Palermo, *4* (1890), 73–88.
2. P. Du Val, *On absolute and non-absolute singularities of algebraic surfaces*, Rev. Fac. Sci. Istanbul (A), *9* (1944), 159–215.
3. ———, *On regular surfaces of genus* 3, Can. J. Math., *3* (1951), 148–154.
4. ———, *On regular surfaces whose canonical system is hyperelliptic*, Can. J. Math., *4* (1952), 204–221.
5. ———, *On loci whose curve sections are hyperelliptic*, J. London Math. Soc., *28* (1953), 1–8.
6. F. Enriques, *Sui sistemi lineari di superficie algebriche ad intersezioni variabili iperellittiche*, Math. Ann., *46* (1895), 179–199.
7. ———, *Le superficie algebriche* (Bologna, 1949).

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A THEOREM ON RINGS

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In a recent paper, Kaplansky [2] proved the following theorem: Let R be a ring with centre Z, and such that $x^{n(x)} \in Z$ for every $x \in R$. If R, in addition, is semi-simple then it is also commutative.

The existence of non-commutative rings in which every element is nilpotent rules out the possibility of extending this result to all rings. One might hope, however, that if R is such that $x^{n(x)} \in Z$ for all $x \in$ R and the nilpotent elements of R are reasonably "well-behaved," then Kaplansky's theorem should be true without the restriction of semi-simplicity.

This is in fact what we obtain in this paper. More precisely, we prove the following two theorems:

THEOREM. *Let R be a ring with centre Z such that $x^{n(x)} \in Z$ for all $x \in R$. Then R is not commutative only if every element in the commutator ideal of R is nilpotent.*

THEOREM. *Let R be a ring with centre Z such that $x^{n(x)} \in Z$ for all $x \in R$. Then if R possesses no non-zero nil-ideals it is commutative.*

Since every nil-ideal of a ring is in the radical of that ring, these results contain that of Kaplansky which we have cited. Any restriction on a ring which will forbid the commutator ideal from being a nil-ideal, in the presence of $x^{n(x)} \in Z$, will force commutativity on the ring in question.

Henceforth every ring R which we consider will have centre Z and the property that $x^{n(x)}$ is in Z for every x in R, where $n(x)$ is a positive integer depending on x. Whenever we use the word ideal we mean a two-sided ideal. We begin with

THEOREM 1. *Suppose that in R,*

(i) *Z possesses no divisors of zero of R,*

(ii) *there is an $a \in Z$, $a \neq 0$, so that, given any non-zero ideal U of R, then for some integer $m(U)$, $a^{m(U)} \in U$. Under these conditions R is commutative.*

Proof. Consider the set of ordered pairs (r, s) where $r \in R$, $s \neq 0 \in Z$. We define $(r_1, s_1) \sim (r_2, s_2)$ if and only if $r_1 s_2 = r_2 s_1$. Clearly this is a proper equivalence relation. We denote the equivalence class of (r, s) by $[r, s]$. Let R^* be the set of all these equivalence classes. In R^* we define an addition and multiplication by

$$[b, c] + [d, g] = [bg + dc, cg] \tag{1}$$

$$[b, c]\,[d, g] = [bd, cg]. \tag{2}$$

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Since Z is free of divisors of zero of R, these operations are meaningful and independent of the particular representatives of the classes. Moreover R^* forms a ring under these operations. If we denote $[rs, s]$ by $[r, 1]$, then the set $\bar{R} = \{[r, 1] \in R^* \mid r \in R\}$ is a ring isomorphic (in the obvious way) to R. Let Z^* be the centre of R^*. A simple computation shows that $[r, s] \in Z^*$ if and only if $r \in Z$. Moreover Z^* is now a *field.*

We wish to show that R^* is a simple ring. Suppose $U^* \neq (0)$ is an ideal of R^*. Let

$$U = \{x \in R \mid [x, 1] \in U^*\}.$$

U is not merely (0), for if $[b, z] \in U^*$, $b \neq 0$, then $[z, 1]\,[b, z] = [b, 1] \in U^*$. A simple verification shows that U is an ideal of R. Since this is so, by our hypothesis (2), $a^{m(u)} \in U$ for an appropriate integer $m(u)$; that is $[a^{m(u)}, 1] \in U^*$. But since $[a^{m(u)}, 1] \in Z^*$ it has an inverse in R^*, whence $U^* = R^*$. Hence R^* is a simple ring, and so is semi-simple. In R^* we also have $x^{n(x)} \in Z^*$ for each $x \in R^*$, and so, by Kaplansky's theorem, R^* is commutative. Since $R^* \supset \bar{R}$ an isomorphic replica of R we immediately have that R is commutative.

We next proceed to

THEOREM 2. *Suppose in R that $c \neq 0$ is an element of the commutator ideal of R. Then if U is any non-zero ideal of R there exists an integer $m(u)$ so that $c^{m(u)} \in U$.*

Proof. Suppose that there exists an ideal U or R such that

$$U \neq (0), \tag{3}$$

$$c^i \notin U \qquad \text{for all } i = 1, 2, \ldots. \tag{4}$$

By Zorn's lemma there exists an ideal V of R possessing the properties (3) and (4) of U and such that if W is any ideal of R with $W \supset V$ and $W \neq V$ then for some integer K, $c^K \in W$. Consider $\bar{R} = R/V$. In $\bar{R}$, from the choice of V, we have

$$\bar{x}^{n(\bar{x})} \in \bar{Z} \qquad \text{for each } \bar{x} \in \bar{R};$$

and if $\bar{a} = \bar{c}^{n(c)} \neq 0 \in \bar{Z}$, then for any non-zero ideal $\bar{T}$ of $\bar{R}$ some power of $\bar{a}$ is in $\bar{T}$.

We claim that there are no divisors of zero of $\bar{R}$ in $\bar{Z}$. For suppose $\bar{z} \in \bar{Z}$ and that $\bar{z}\bar{x} = 0$, $\bar{z} \neq 0 \neq \bar{x}$. Let

$$A(\bar{z}) = \{\bar{x} \in \bar{R} \mid \bar{z}\bar{x} = 0\}.$$

Clearly $A(\bar{z})$ is an ideal of R and is not the zero ideal, hence $\bar{a}^i \in A(\bar{z})$ for an appropriate i. Thus $za^i \in V$ where $\bar{z} = z + V$, $\bar{a} = a + V$. Without loss of generality $i = n(a)^t$ for large enough t. Let

$$T = \{y \in R \mid ya^i \in V\}.$$

Since $a^i \in Z$, T is an ideal of R, and clearly $T \supset V$. If $T \neq V$ then $c^j \in T$ for appropriate j, whence a^K is in T for appropriate K; thus $a^{i+K} \in V$ and so

some power of c is in V, a contradiction. Hence $\bar{Z}$ has no zero-divisors of $\bar{R}$. But then all the conditions of Theorem 1 are fulfilled, so $\bar{R} = R/V$ is commutative. Hence $V \supset$ commutator ideal $\supset c$, a contradiction, and Theorem 2 is established.

We are now led to

THEOREM 3. *Suppose $0 \neq c$ is in the commutator ideal of R and that c is not nilpotent. Then c is not a divisor of zero.*

Proof. Let $a = c^{n(c)} \neq 0$; $a \in Z$, and suppose that $cx = 0$, $x \neq 0$. Then certainly $ax = 0$. Let

$$A(a) = \{x \in R \mid ax = 0\}.$$

Since $a \in Z$, $A(a)$ is an ideal of R and $A(a) \neq (0)$. So by Theorem 2, $c^t \in A(a)$ for a suitable t, and so $a^j \in A(a)$ for some j; then $a^{j+1} = 0$, forcing a, and so c, to be nilpotent, contradicting our hypothesis. Thus Theorem 3 is established.

We are now in a position where we can prove the main theorem of the paper, namely,

THEOREM 4. *Suppose R is a ring with centre Z and such that $x^{n(x)} \in Z$ for all $x \in R$. Then if R is not commutative, the commutator ideal of R is a non-zero nil-ideal.*

Proof. Suppose that R is not commutative, and that $c \neq 0$ is in the commutator ideal of R and is not nilpotent. By Theorem 3, c is not a divisor of zero. Suppose there is a $z \neq 0$ in Z which is a divisor of zero, say $zx = 0$, $x \neq 0$. Let $A(z) = \{x \in R \mid zx = 0\}$. $A(z)$ is an ideal of R and is not (0). So by Theorem 2, $c^i \in A(z)$ for some i. But then $c^i z = 0$, whence $z = 0$ since c is not a zero-divisor. So no $0 \neq z \in Z$ is a divisor of zero of R. Let us list the properties of R:

(a) $x^{n(x)} \in Z$ for all $x \in R$.

(b) Z has no divisors of zero of R.

(c) There exists a $b \in Z$ which is not nilpotent such that given any non-zero ideal U of R then $b^{m(u)} \in U$ (by the above remarks, $b = c^{n(c)}$ will do if c is any non-nilpotent element of the commutator ideal).

Thus all the conditions of Theorem 1 are satisfied, and so R is commutative, contrary to our assumption. Hence we are forced to conclude that every element in the commutator ideal is nilpotent, proving Theorem 4.

THEOREM 5. *Suppose the ring R is such that $x^{n(x)} \in Z$, the centre of R, for all $x \in R$. Then if R has no non-zero nil-ideals, it must be commutative.*

Theorem 5 is an immediate consequence of Theorem 4, but actually the two results are equivalent. For suppose R is a ring with non-zero nil-ideals, then by a result of Köthe [**3**] the sum of all nil-ideals of R is a nil-ideal T, R/T possesses no non-zero nil-ideals, and in R/T, $x^{n(x)}$ is in the centre. So by Theorem 5, R/T

is commutative; hence $T \supset$ commutator ideal, which thus must be a nil-ideal; consequently Theorem 5 implies Theorem 4.

It might be pointed out that Theorem 5 cannot be appreciably weakened. The only plausible weakening would be to change "no nil-ideals" to "no nilpotent ideals" in the statement of Theorem 5, but there is an example, due to Baer [**1**], of a nil-ring with no nilpotent ideals; this rules out the possibility of the stronger result.

References

1. Reinhold Baer, *Radical ideals*, Amer. J. Math., *65* (1943), 540.
2. Irving Kaplansky, *A theorem on division rings*, Can. J. Math., *3* (1951), 290-292.
3. G. Köthe, *Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist*, Math. Z., *32* (1930), 170.

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ON THE COMMUTATIVITY OF CERTAIN DIVISION RINGS

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Formerly Hua [1] proved that if A is a division ring with centre Z and if there exists a natural number n such that $a^n \in Z$ for every $a \in A$, then A is commutative; this generalizes Wedderburn's theorem on finite division rings. Another generalization of Wedderburn's theorem, due to Jacobson [3], asserts that every algebraic division algebra over a finite field is commutative. On the other hand, a theorem of Noether and Jacboson [3] states that every non-commutative algebraic division algebra contains an element which is not contained in the centre Z and is separable over Z. These results have been successfully unified by Kaplansky [4] into one theorem, which asserts that if there exists for every element a in a division ring A with centre Z a natural number $n(a)$ (depending, perhaps, on a) such that $a^{n(a)} \in Z$, then A is commutative. He also proved that if there exists a (fixed) non-zero polynomial f with coefficients in Z and without constant term, such that $f(a) \in Z$ for every $a \in A$, then A is commutative. Recently Ikeda [2] obtained a certain generalization of the former of these theorems of Kaplansky, which deals with polynomials with coefficients from the prime field, instead of single powers, and which includes a particular case of the latter of Kaplansky's theorems. In the present note we prove the following theorem[1] which includes all these results:

THEOREM. *Let A be a division ring and Z be its centre. Let r be a natural number and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be r (fixed) non-zero elements in Z. Suppose that there exist, for each element a of A, r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ such that*

$$n_1(a) < n_i(a) \qquad (i = 2, \ldots, r), \tag{1}$$

$$a^{n_1(a)}\alpha_1 + a^{n_2(a)}\alpha_2 + \ldots + a^{n_r(a)}\alpha_r \in Z. \tag{2}$$

Then necessarily $A = Z$, that is, A is commutative.

Our proof is somewhat arithmetical (in a weak sense), while the approaches of the former authors have all been algebraic. We need

LEMMA 1. *Let Z be a field which is either*

(i) *of characteristic* 0, *or*

(ii) *of characteristic $p \neq 0$ and non-algebraic over its prime field, and let L be an algebraic proper extension of Z which is not purely inseparable over Z. Then*

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[1] Cf. I. N. Herstein, *A generalization of a theorem of Jacobson* III, Amer. J. Math., *75* (1953), 105–111.

there exists a pair of distinct (special) exponential valuations ρ_1, ρ_2 in L which coincide on Z.

This lemma is perhaps more or less known; anyway we shall come back to it elsewhere.

LEMMA 2. *Let Z and L be as in Lemma 1. There cannot exist a natural number r and a set of r non-zero elements $\alpha_1, \alpha_2, \ldots, \alpha_r$ in L such that for each element a in L there exist r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ satisfying the conditions* (1) *and* (2) *in our Theorem.*

Proof. Let r be a natural number, and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be r non-zero elements in L. Let ρ_1, ρ_2 be as in Lemma 1. Take two elements a_1, a_2 in L such that

$$\rho_1(a_1) \gg 1, \quad \rho_2(a_1) = 0, \quad \rho_1(a_2) = 0, \quad \rho_2(a_2) \gg 1.$$

Let k be a natural number larger than all of $2|\rho_j(\alpha_i)|$ $(i = 1, 2, \ldots, r; j = 1, 2)$, and let m be a natural number such that $m\rho_2(a_2) - \rho_1(a_1) > 1$. Put $a = a_1^k a_2^{mk}$.

Let, now, $n_1(a), n_2(a), \ldots, n_r(a)$ be r natural numbers satisfying (1), and consider the sum $\sum a^{n_i(a)}\alpha_i$. Observing (1) and $\rho_1(a) \gg k$, we see readily that the ρ_1-value of the sum is simply the ρ_1-value of its first term, i.e.

$$\rho_1(a^{n_1(a)}\alpha_1) = n_1(a)k\rho_1(a_1) + \rho_1(\alpha_1). \tag{3}$$

Similarly the ρ_2-value of the same sum is equal to

$$\rho_2(a^{n_1(a)}\alpha_1) = n_1(a)mk\rho_2(a_2) + \rho_2(\alpha_1). \tag{4}$$

These two numbers (3) and (4) are not equal. For, if they were equal, then

$$m\rho_2(a_2) - \rho_1(a_1) = (\rho_1(\alpha_1) - \rho_2(\alpha_1))/n_1(a)k < 1$$

contrary to our choice of m. Thus our sum

$$\sum a^{n_i(a)}\alpha_i$$

cannot belong to Z. The lemma is thus proved.[2]

Now we can derive our Theorem exactly as in Kaplansky [**4**]. Thus suppose that $A \neq Z$, and let a be an element of A not contained in Z and separable over Z (Theorem of Noether and Jacboson). Let L be the field generated by a over Z. It follows from Lemma 2 that Z must be of characteristic $p \neq 0$ and algebraic over its prime field. But this is a contradiction, by virtue of the first cited theorem of Wedderburn-Jacobson.

Theorem 7 of Hua [**1**] actually states that a non-commutative division ring is generated by the nth powers of its elements, n being an arbitrary natural number. Also Kaplansky [**4**] gives a corresponding modification of his result. by means of a theorem of Cartan-Brauer-Hua [**1**, Theorem 2]. Our theorem too may be combined with the Cartan-Brauer-Hua theorem, to yield

[2]Cf. [**5**], setting $\alpha_1 = -1, \alpha_2 = 1, \alpha_3 = \ldots = \alpha_r = 0$..

COROLLARY. *Let A be a non-commutative division ring and Z be its centre. Let r be a natural number, and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a set of r non-zero elements in Z. Let there be given for each element a in A a set of r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ such that $n_1(a) < n_i(a)$ $(i = 2, \ldots, r)$ and*

$$n_i(a) = n_i(c^{-1}ac) \qquad (i = 1, 2, \ldots, r),$$

for every non-zero element c in A. Then A is generated, as a division ring, by the elements

$$\sum a^{n_i(a)}\alpha_i ,$$

where a runs over A.

Added March 28, 1953. After the submission of the present note for publication I obtained access to the papers by Herstein (referred to in footnote 1) and Krasner [**5**] where a valuation-theoretical approach, analogous to ours, is made in similar context. Krasner's theorem is a particular case of our Lemma 2, while the division ring case of Herstein's result is a special case of our Theorem. As to our Lemma 1, a simple proof (which yields in fact a little more) will be given in M. Nagata, T. Nakayama, and T. Tuzuku, *An existence lemma in valuation theory*, to appear in the Nagoya Mathematical Journal.

REFERENCES

1. L. K. Hua, *Some properties of a sfield*, Proc. Nat. Acad. Sci. U.S.A., *35* (1949), 533–537.
2. M. Ikeda, *On a theorem of Kaplansky*, Osaka Math. J., *4* (1952), 235–240.
3. N. Jacobson, *Structure theory of algebraic algebras of bounded degree*, Ann. Math., *46* (1945), 695–707.
4. I. Kaplansky, *A theorem on division rings*, Can. J. Math., *3* (1951), 290–292.
5. M. Krasner, *On the non-existence of certain extensions*, Amer. J. Math., *75* (1953), 112–116.

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ON NON-AVERAGING SETS OF INTEGERS

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1. Introduction. Let S be a set of positive integers no three of which are in arithmetical progression, i.e., if A, B, C are distinct elements of S, $A + B \neq 2C$. We call such a set a non-averaging set. Let $\nu(n)$ denote the maximum number of elements not exceeding n in any non-averaging set. The problem of finding bounds for $\nu(n)$ has been treated by several authors [**1, 3, 5, 6, 7**]. The question first arose in connection with a theorem of van der Waerden [**8**]. Van der Waerden's theorem states that if one separates the integers $1, 2, 3, \ldots, N$ into k disjoint classes, then for every l, there exists at least one class which contains an arithmetical progression of l terms, if $N = (k, l)$ is sufficiently large. This theorem was used by Brauer [**2**] to prove the existence of sequences of l consecutive quadratic residues and l consecutive non-residues for every sufficiently large prime. In van der Waerden's theorem the $N(k, l)$ is extremely large, and it was thought that a study of $\nu(n)$ would yield better bounds for N. Unfortunately this hope has not as yet been fulfilled.

G. Szekeres conjectured that $\nu\{(3^k + 1)/2\} = 2^k$ and this was proved [**3**] for $k < 5$. This would make

$$\nu(n) < cn^{\log 2/\log 3}$$

for some fixed c. The conjecture was proved false by Salem and Spencer [**6**] who showed that for every $\epsilon > 0$ and sufficiently large n,

$$\nu(n) > n^{1-(\log 2+\epsilon)/(\log\log n)}. \tag{1.1}$$

This result was refined by Behrend [**1**] who proved that for $\epsilon > 0$ and sufficiently large n,

$$\nu(n) > n^{1-(2\sqrt{(2\log 2)}+\epsilon)/\sqrt{(\log n)}}. \tag{1.2}$$

In Behrend's method the set S depends upon n, i.e., the set used for $n = 1000$ might be quite different from that for $n = 1001$. Furthermore, the argument makes use of the Dirichlet's principle of drawers and hence is not constructive. In §2 we give a constructive definition of an infinite sequence R which has no three terms in arithmetic progression, and which yields, for n sufficiently large,

$$\nu(n) > n^{1-c/\sqrt{(\log n)}}, \tag{1.3}$$

where c is a fixed constant.

P. Erdös and P. Turán gave some upper bounds for $\nu(n)$ in [**3**]. They proved that for $n > 8$,

$$\nu(2n) < n. \tag{1.4}$$

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Also for every $\epsilon > 0$ and n sufficiently large they proved

$$\nu(n) < (\tfrac{4}{9} + \epsilon)n. \tag{1.5}$$

Finally they stated without proof that

$$\nu(n) < (\tfrac{3}{8} + \epsilon)n. \tag{1.6}$$

In §3 we shall use a different method to prove

$$\nu(n) < \tfrac{2}{5}n + 3. \tag{1.7}$$

This has the advantage of being free of ϵ. We shall then use the same method but with a much longer argument to prove that

$$\nu(n) < \tfrac{4}{11}n + 5, \tag{1.8}$$

which is stronger than (1.6).

It has long been conjectured that $\nu(n) = o(n)$, but this has only recently been proved [**5**]. In this connection it is interesting to note the following theorem proved by Redheffer [**4**]. A necessary and sufficient condition that every non-averaging set $\{\lambda_n\}$ have zero density is that

$$\{e^{i\lambda_n x}\}$$

be always incomplete on every interval.

2. Lower bounds for $\nu(n)$. We shall now define an infinite sequence R, show that no three of its elements are in arithmetical progression, and show that if $\nu^*(n)$ denotes the number of elements in R and not exceeding n, then

$$\nu^*(n) > n^{1-c/\sqrt{(\log n)}},$$

where c is a fixed constant.

Given a number x, written in the denary scale, we decide whether x is in R on the basis of the following rules:

First we enclose x in a set of brackets, putting the first digit (counting from right to left) in the first bracket, the next two in the second bracket, the next three in the third bracket, and so on. If the last non-empty bracket (the bracket furthest to the left which does not consist entirely of zeros) does not have a maximal number of digits, we fill it with zeros. For instance, the numbers

$$A = 32653200200, \quad B = 100026100150600, \quad C = 100086600290500$$

would be bracketed as follows:

$$A = (00003)\ (2653)\ (200)\ (20)\ (0), \qquad B = (10002)\ (6100)\ (150)\ (60)\ (0),$$
$$C = (10008)\ (6600)\ (290)\ (50)\ (0).$$

Now suppose the rth bracket in x contains non-zero digits, but all further brackets to the left are zero. Call the number represented by the digits in the ith bracket x_i, $i = 1, 2, \ldots, r-2$. Further, denote by $\bar{x}$ the number represented

by the digits in the last two brackets taken together, but excluding the last digit. For x to belong to R we require:

(2.1) The last digit of x must be 1;

(2.2) x_i must begin with 0 for $i = 1, 2, \ldots, r-2$;

(2.3) $$\sum_{i=1}^{r-2} x_i^2 = \bar{x}.$$

In particular, we note that A satisfies (2.2) but violates (2.1) and (2.3) and so is not in R, but B and C satisfy all three conditions and so are in R. To check (2.3) for B we note that $60^2 + 150^2 = 26100$.

We next prove that no three integers of R are in arithmetic progression. First note that if two elements of R have a different number of non-empty brackets then their arithmetic mean cannot satisfy (2.1). Thus we need only consider averages of elements of R having the same number of non-empty brackets. From conditions (2.1) and (2.3) it follows that two elements of R can be averaged bracket by bracket for the first $r-2$ brackets and also for the last two brackets taken together. Thus in our example

$$\tfrac{1}{2}(60 + 50) = 55, \quad \tfrac{1}{2}(150 + 290) = 220,$$

$$\tfrac{1}{2}(100026100 + 100086600) = 100056350,$$

$$\tfrac{1}{2}(B + C) = (10005)\ (6350)(220)(55)(0).$$

This violates (2.3) and so cannot be in R. In general we will prove that if x and y are in R, then $z = \frac{1}{2}(x + y)$ will violate (2.3).

Since x and y are in R,

$$\bar{z} = \frac{\bar{x} + \bar{y}}{2} = \sum_{i=1}^{r-2} \frac{x_i^2 + y_i^2}{2}.$$

On the other hand z in R implies

$$\bar{z} = \sum_{i=1}^{r-2} z_i^2 = \sum_{i=1}^{r-2} \left(\frac{x_i + y_i}{2}\right)^2.$$

Hence if z is in R then

$$\sum_{i=1}^{r-2} \frac{x_i^2 + y_i^2}{2} = \sum_{i=1}^{r-2} \left(\frac{x_i + y_i}{2}\right)^2.$$

Thus

$$\sum_{i=1}^{r-2} \left(\frac{x_1 - y_i}{2}\right)^2 = 0,$$

which implies $x_i = y_i$ for $i = 1, 2, \ldots, r-2$. This together with (2.1) and (2.2) implies that x and y are not distinct.

Szekeres' sequence starts with 1, 2, 4, 5, 10, 11, 14, 28, 29, ... Our sequence starts with

100000, 1000100100, 1000400200, 100250500, ...

Nevertheless, it will be proved that the terms of this sequence eventually become smaller than the corresponding terms of the first sequence.

The first sequence is maximal in the sense that no number can be added to it without introducing an arithmetical progression of three terms. Our sequence is not maximal in this sense and indeed it is easy to find numbers, among them 1, which may be adjoined. However, there seems little point in doing this since it improves (1.3) only by an ϵ in the constant.

We now estimate how many integers in R contain exactly r brackets. Given r brackets we can make the first digit in each of the first $r-2$ brackets 0. We then fill up the first $r-2$ brackets in an arbitary manner. This can be done in

$$10^{0+1+2+\dots+(r-2)} = 10^{\frac{1}{2}(r-2)(r-3)}$$

ways. The last two brackets can then be filled in such a way as to satisfy (2.1) and (2.3). To see this we need only check that the last two brackets will not be overfilled, and that the last digit, which we shall set equal to 1, will not be interfered with. This follows from the inequality

$$(10^1)^2 + (10^2)^2 + \dots + (10^{r-2})^2 < 10^{2(r-1)}.$$

For a given n let r be the integer determined by

$$10^{\frac{1}{2}r(r+2)} \leqslant n < 10^{\frac{1}{2}(r+1)(r+2)}. \tag{2.4}$$

Since all the integers with at most r brackets will not exceed n, and since r brackets can be filled to specification in $10^{\frac{1}{2}(r-3)(r-2)}$ ways, we have

$$\nu^*(n) \geqslant 10^{\frac{1}{2}(r-2)(r-3)}. \tag{2.5}$$

From the right-hand side of (2.4) we obtain, using logarithms to base 10,

$$r + 2 > \sqrt{(2 \log n)}$$

so that (2.5) implies, for sufficiently large n,

$$\nu^*(n) \geqslant 10^{\frac{1}{2}(r-2)(r-3)} > 10^{\log n - 9\sqrt{(2 \log n)}} > 10^{\log n(1-c/\sqrt{(\log n)})} = n^{1-c/\sqrt{(\log n)}}.$$

This proves (1.3) since $\nu(n) \geqslant \nu^*(n)$.

Use of base 2 instead of base 10, together with a more refined treatment of the inequalities, would yield a better value for the constant.

3. Upper bounds for $\nu(n)$. Let $a_1 < a_2 < \dots < a_r$ denote the even elements, and by $b_1 < b_2 < \dots < b_s$ the odd elements of S, not exceeding n. Denote the integers $\frac{1}{2}(a_i + a_j)$ by (i, j). It follows from the definition of S in §1 that (i, j) is not in S.

Now denote the following set of $2r-3$ integers by $\bar{S}$:

$$(1, 2) < (1, 3) < (2, 3) < (2, 4) < \dots < (i, i+1) < (i, i+2) < \dots$$
$$< (r-2, r-1) < (r-2, r) < (r-1, r).$$

The integers of $\bar{S}$ are clearly under n and not in S. Furthermore, we shall see that at least one of the integers $(1, 4)$ and $(1, 5)$ which are not in S, is also not in $\bar{S}$.

To see this note that if (1, 4) is in $\bar{S}$, then (1, 4) = (2, 3). If also (1, 5) is in $\bar{S}$, then either (1, 5) = (2, 3) or (1, 5) = (2, 4), or (1, 5) = (3, 4). Now (1, 4) = (2, 3) together with (1, 5) = (2, 3) implies $a_4 = a_5$. Also (1, 4) = (2, 3) together with (1, 5) = (2, 4) implies $a_3 + a_5 = 2a_4$. Finally, (1, 4) = (2, 3) together with (1, 5) = (2, 4) implies $a_3 + a_5 = 2a_4$. Similarly, at least one of each pair

$$(i, i+3), \quad (i, i+4), \qquad i = 1, 2, \ldots, r-4$$

must be absent from S and differ from any in $\bar{S}$. Thus we have at least $(2r-3) + (r-4) = 3r-7$ numbers under n and not in S. Present in S are exactly $r+s$ numbers. Hence we have

$$3r - 7 + r + s \leqslant n. \tag{3.1}$$

Let us assume that $r \geqslant s$. Then (3.1) yields

$$\tfrac{3}{2}(r+s) - 7 + (r+s) \leqslant n; \tag{3.2}$$

and since $\nu(n) = r + s$, this yields

$$\nu(n) < \tfrac{2}{5}n + 3.$$

If $r < s$ then we can deal with the b's instead of the a's and the same conclusion will follow. Thus in any case (1.5) is proved.

Our final object is to prove

$$\nu(n) < \tfrac{4}{11}n + 5.$$

We use the same notation as before. Again we have the $2r - 3$ integers of $\bar{S}$ not in S. This time however we will prove that of the integers

$$(1, 4), (1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7),$$

which we will denote by T (and which are not in S) at least three distinct ones are not in $\bar{S}$

The argument will then closely follow the lines used in the previous theorem. We will first prove the following

LEMMA. *Two different equations of the type* $(i_1, i_2) = (i_3, i_4)$ *involving only five distinct integers under* r, *imply a contradiction.*

Proof. Let the five different numbers involved be $i_1 < i_2 < i_3 < i_4 < i_5$. Clearly the only possible equations of the required type involving these are:

A: $(i_1, i_4) = (i_2, i_3)$ B: $(i_1, i_5) = (i_2, i_3)$ C: $(i_1, i_5) = (i_2, i_4)$
D: $(i_1, i_5) = (i_3, i_4)$ E: $(i_2, i_5) = (i_3, i_4)$

But now A together with B implies $a_{i_4} = a_{i_5}$; A together with C implies

$$a_{i_3} + a_{i_5} = 2a_{i_4}.$$

Similarly each of the other eight combinations easily leads to a contradiction.

We shall now prove that at least three distinct numbers in T are not in $\bar{S}$. We consider three cases.

Case 1. Suppose that (2, 5) is in $\bar{S}$. Now (2, 5) in $\bar{S}$ implies (2, 5) = (3, 4).

Note that we need not concern ourselves here with possibilities like $(2, 5) = (1, i)$ since then $i > 5$ and $(1, i)$ will not be in $\bar{S}$. If $(1, 4)$ is in $\bar{S}$, then $(1, 4) = (2, 3)$ which by the lemma is incompatible with $(2, 5) = (3, 4)$. Hence $(1, 4)$ is not in $\bar{S}$.

If $(1, 5)$ is in $\bar{S}$ then we have one of

$$(1, 5) = (2, 5), \quad (1, 5) = (2, 4), \quad (1, 5) = (3, 4)$$

but each of these possibilities is incompatible with $(2, 5) = (3, 4)$ by the lemma. Hence $(1, 5)$ is not in $\bar{S}$. If $(2, 6)$ is in $\bar{S}$, then we have one of

$$(2, 6) = (3, 4), \quad (2, 6) = (3, 5), \quad (2, 6) = (4, 5),$$

but each of these is incompatible with $(2, 5) = (3, 4)$ by the lemma. Hence $(2, 6)$ is not in $\bar{S}$.

Thus, in case (1) the three numbers $(1, 4)$ $(1, 5)$ $(2, 6)$ are not in $\bar{S}$ and we are finished. We may therefore assume $(2, 5)$ is not in $\bar{S}$, and need only show that two other numbers of T are not in $\bar{S}$.

Case 2. Suppose $(1, 4)$ is in $\bar{S}$. Now $(1, 4)$ in $\bar{S}$ implies $(1, 4) = (2, 3)$ and $(1, 5)$ in $\bar{S}$ implies one of

$$(1, 5) = (2, 3), \quad (1, 5) = (2, 4), \quad (1, 5) = (3, 4)$$

each of which is incompatible with $(1, 4) = (2, 3)$ by the lemma. Hence $(1, 5)$ is not in $\bar{S}$, and it will suffice to show that at least one other number in T is not in $\bar{S}$. If this be false then we must have at least one inequality in each of the following columns:

A: $(1, 6) = (2, 3)$	A: $(2, 6) = (3, 4)$	A: $(2, 7) = (3, 4)$
A: $(1, 6) = (2, 4)$	C: $(2, 6) = (3, 5)$	$(2, 7) = (3, 5)$
A: $(1, 6) = (3, 4)$	$(2, 6) = (4, 5)$	D: $(2, 7) = (4, 5)$
$(1, 6) = (3, 5)$		D: $(2, 7) = (4, 6)$
B: $(1, 6) = (4, 5)$		D: $(2, 7) = (5, 6)$

The possibilities marked A are out by $(1, 4) = (2, 3)$ and the lemma. B: $(1, 6) = (4, 5)$ is out for then no possibility for $(2, 6)$ makes $(2, 6) > (1, 6)$. Now we have $(1, 6) = (3, 5)$ and this with the lemma eliminates the possibility marked C. Now $(2, 6) = (4, 5)$ so that by the lemma the possibilities marked D are out. But now the remaining possibilities for $(2,7)$, namely $(2, 7) = (3, 5)$ makes $(2, 7) < (2, 6)$ which is impossible. Hence in this case we are finished, and so we may assume $(1, 4)$ is not in $\bar{S}$.

Case 3. We have already $(1, 4)$ and $(2, 5)$ not in $\bar{S}$. Hence it will suffice to show that at least one other number is in T and not in $\bar{S}$. If this be false we must have at least one equality in each of the following columns:

G: $(1, 5) = (2, 3)$	A: $(1, 6) = (2, 3)$	D: $(2, 6) = (3, 4)$
$(1, 5) = (2, 4)$	B: $(1, 6) = (2, 4)$	$(2, 6) = (3, 5)$
E: $(1, 5) = (3, 4)$	$(1, 6) = (3, 4)$	F: $(2, 6) = (4, 5)$
	C: $(1. 6) = (3, 5)$	
	A: $(1, 6) = (4, 5)$	

D: $(1, 7) = (2, 3)$	
D: $(1, 7) = (2, 4)$	A: $(2, 7) = (3, 4)$
D: $(1, 7) = (3, 4)$	G: $(2, 7) = (3, 5)$
$(1, 7) = (3, 5)$	H: $(2, 7) = (4, 5)$
H: $(1, 7) = (4, 5)$	$(2, 7) = (4, 6)$
E: $(1, 7) = (4, 6)$	G: $(2, 7) = (5, 6)$
A: $(1, 7) = (5, 6)$	

Now $(1, 6) = (2, 3)$ would not leave any consistent possibility for $(1, 5)$ while $(1, 6) = (4, 5)$ would not leave any consistent possibility for $(2, 6)$. Similarly $(1, 7) = (5, 6)$ and $(2, 7) = (3, 4)$ are out. These four possibilities are marked A.

If B: $(1, 6) = (2, 4)$ then by the lemma, the only possibility for $(2, 6)$ is $(2, 6) = (3, 5)$.

Also $(1, 6) = (2, 4)$ and $(1, 6) > (1, 5)$ gives as only possibility for $(1, 5)$, $(1, 5) = (2, 3)$.

Now $(2, 6) = (3, 5)$ and $(1, 5) = (2, 3)$ are incompatible by the lemma Hence B is out.

If C: $(1, 6) = (3, 5)$ then by the lemma, the only possibility for $(1, 5)$ is $(1, 5) = (2, 4)$.

Also $(1, 6) = (3, 5)$ and $(2, 6) > (1, 6)$ gives as only possibility for $(2, 6)$, $(2, 6) = (4, 5)$.

Now $(1, 5) = (2, 4)$ and $(2, 6) = (4, 5)$ are incompatible by the lemma; hence C is out.

We now have $(1, 6) = (3, 4)$, and since $(2, 7) > (1, 7) > (1, 6)$ and $(2, 6) > (1, 6)$ the possibilities marked D are out. Furthermore, by the lemma $(1, 6) = (3, 4)$ eliminates the possibilities marked E.

Now F: $(2, 6) = (4, 5)$ leaves every choice for $(2, 7)$ either not larger than $(2, 6)$ or incompatible with $(2, 6) = (4, 5)$ by the lemma; hence F is out.

We now have $(2, 6) = (3, 5)$ which by the lemma eliminates the possibilities marked G.

This leaves $(1, 5) = (2, 4)$ which by the lemma eliminates the possibilities marked H.

We thus have $(2, 6) = (3, 5) = (1, 7)$ but $(2, 6) = (1, 7)$ eliminates the remaining possibilities for $(2, 7)$. Hence case 3 is complete.

The theorem which we have thus proved for the set T:

$$(1, 4),\ (1, 5),\ (1, 6),\ (1, 7),\ (2, 5),\ (2, 6),\ (2, 7)$$

will go through with only an obvious change in notation for the set T_i:

$$(i,\ i+3),\ (i,\ i+4),\ (i,\ i+5),\ (i,\ i+6)\ (i+1,\ i+4),\ (i+1,\ i+5),\ (i+1, i+6), \qquad i = 1, 2, \ldots, r-6.$$

Furthermore, the sets T_i are disjoint for $i = 1, 3, 5, \ldots, k$ where k is the largest odd integer in $r - 6$. Certainly then we may take $k > \frac{1}{2}(r - 6)$.

Thus we have at least $\frac{3}{2}$ $(r - 6)$ integers not in S and not in $\bar{S}$. Altogether

then, we have

$$(2r - 3) + \tfrac{3}{2}(r - 6) = \tfrac{7}{2}r - 12$$

integers not in S.

In S we have exactly $r + s$ integers, hence

$$\tfrac{7}{2}r - 12 + r + s < n.$$

Assuming $r > s$ gives

$$\tfrac{7}{4}(r + s) + (r + s) - 12 = \tfrac{11}{4}(r + s) - 12 < n;$$

and since $\nu(n) = r + s$ we have

$$\nu(n) < \tfrac{4}{11}(n + 12) < \tfrac{4}{11}n + 5.$$

On the other hand, if $r < s$, the same result can be obtained by working with the b's instead of the a's.

References

1. F. A. Behrend, *On sets of integers which contain no three in arithmetical progression*, Proc. Nat. Acad. Sci., *23* (1946), 331–332.
2. A. Brauer, *Über Sequenzen von Potenzresten*, S.B. preuss. Akad. Wiss. (1928), 9–16.
3. P. Erdös and P. Turán, *On some sequences of integers*, Proc. London Math. Soc. (2), *11* (1936), 261–264.
4. R. M. Redheffer, *Remarks on incompleteness of* $\{e^{\lambda_n z}\}$, *non-averaging sets, and entire functions*, Proc. Amer. Math. Soc., *2* (1951), 365–369.
5. M. K. Roth, *Sur quelques ensembles d'entiers*, C.R. Acad. Sci., Paris, *234* (1952), 388–390.
6. R. Salem and D. C. Spencer, *On sets of integers which contain no three in arithmetical progression*, Proc. Nat. Acad. Sci., *28* (1942), 561–563.
7. ———, *On sets which do not contain a given number in arithmetical progression*, Nieuw Arch. Wiskunde (2), *23* (1950), 133–143.
8. B. L. van der Waerden, *Beweis einer Baudetischen Vermutung*, Nieuw Arch. Wiskunde (2), *15* (1927), 212–216.

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A CRITERION FOR IRRATIONALITY

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1. Introduction. The question of the irrationality of functions defined by power series, for rational values of the variable, has attracted much attention for over a hundred years. Legendre, in generalizing Lambert's proof of the irrationality of tan x for rational x, proved an important theorem on the irrationality of continued fractions with integer elements. Here we use Legendre's theorem (Lemma 3) to prove that at least one of a certain pair of power series is irrational whenever the variable is rational and satisfies a further condition.

We prove the following:

THEOREM 1. *Let $\psi(n)$ be any positive, integral-valued strictly increasing function of n. Let*

$$F(x) = \sum_{p=0}^{\infty} a_p x^p, \quad G(x) = \sum_{p=0}^{\infty} b_p x^p,$$

where a_p is the number of partitions of p of the form

$$p = \psi(m_1) + \psi(m_2) + \ldots + \psi(m_k); \quad m_k > m_{k-1} > \ldots > m_1 = 1;$$
$$m_{i+1} - m_i \geq 2;$$

$b_0 = 1$, and for $p \geq 1$, b_p is the number of partitions of p of the form

$$p = \psi(m_1) + \psi(m_2) + \ldots + \psi(m_k); \quad m_k > m_{k-1} > \ldots > m_1 \geq 2;$$
$$m_{i+1} - m_i \geq 2.$$

Let

$$\gamma = \liminf_{n \to \infty} \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r).$$

Then if r, s are positive integers such that $(r, s) = 1$ and $r < s^{\gamma}$, the number

$$H(r/s) = F(r/s)/G(r/s)$$

is irrational.

2. Subsidiary results.

LEMMA 1. *$F(x)$ and $G(x)$ converge if $|x| < 1$.*

Proof. Since $\psi(n)$ is positive, integral-valued, and strictly increasing, $\psi(1) \geq 1, \psi(2) \geq 2, \ldots, \psi(n) \geq n$. Hence $a_n, b_n \leq p(n)$ where $p(n)$ denotes the number of partitions of n into the sum of positive integers. From Euler's product

$$1 + \sum_{n=1}^{\infty} p(n) x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

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for $|x| < 1$, it follows that $F(x)$, $G(x)$ converge if $|x| < 1$. Henceforth we suppose that $0 < x < 1$.

LEMMA 2. *The function* $H(x) = F(x)/G(x)$ *has the continued fraction expansion*

$$H(x) = \frac{x^{\psi(1)}}{1+}\ \frac{x^{\psi(2)}}{1+}\ \frac{x^{\psi(3)}}{1+}\cdots.$$

Proof. Let $p_n(x)/q_n(x)$ denote the nth convergent to the continued fraction. Then, for $n \geqslant 3$:

$$\begin{aligned} p_n(x) &= x^{\psi(n)} p_{n-2}(x) + p_{n-1}(x), \\ q_n(x) &= x^{\psi(n)} q_{n-2}(x) + q_{n-1}(x), \end{aligned} \tag{1}$$

subject to:

$$\begin{aligned} p_1(x) &= p_2(x) = x^{\psi(1)}, \\ q_1(x) &= 1; \quad q_2(x) = 1 + x^{\psi(2)}. \end{aligned} \tag{2}$$

From (1) and (2) we see that $p_n(x)$ and $q_n(x)$ have no common factor. If $a_{r,n}$ denotes the coefficient of x^r in $p_n(x)$ it follows from (1) that

$$a_{r,n} = a_{r-\psi(n),n-2} + a_{r,n-1}\,. \tag{3}$$

Since the $a_{p,q}$ are clearly non-negative, (3) implies that

$$a_{r,n} \geqslant a_{r,n-1}\,. \tag{4}$$

On the other hand, if n_1 is that positive integer defined uniquely by the inequality

$$\psi(n_1) \leqslant r < \psi(n_1 + 1)\,, \tag{5}$$

on using (3), with $n_1 + 1$ in place of n, and noting that $a_{s,n} = 0$ if $s < 0$, we deduce that

$$a_{r,n_1+1} = a_{r,n_1}\,,$$

and in fact that

$$\begin{aligned} a_{r,n} &\leqslant a_{r,n+1}\,, & n &< n_1, \\ a_{r,n} &= a_{r,n+1}\,, & n &\geqslant n_1. \end{aligned} \tag{6}$$

Write

$$p_n(x) = \sum_{r=0}^{N} a_{r,n}\, x^r, \qquad N = N(n).$$

Repeated application of the recurrence formula (1) gives

$$p_n(x) = \theta_1(x)\, p_1(x) + \theta_2(x)\, p_2(x)\,,$$

where $\theta_1(x)$, $\theta_2(x)$ are polynomials in x. Since, from (2), $p_1(x) = p_2(x) = x^{\psi(1)}$ it follows that $x^{\psi(1)}$ is a factor of $p_n(x)$. This and the recurrence formula show that $a_{r,n}$ is equal to the number of decompositions of r in the form

$$r = \sum_{i=1}^{p} \psi(m_i)\,,$$

where the m_i are positive integers satisfying

$$1 = m_1 < m_2 < \ldots < m_p \leqslant n, \quad m_{i+1} - m_i \geqslant 2.$$

Conversely, any decomposition of this type will contribute just to the coefficient of x^r in $p_n(x)$ and so $a_{r,n}$ is equal to the number of such decompositions. Then for $n \geqslant n_1$ defined by (5),

$$a_{r,n_1} = a_{r,n_1+1} = \ldots = a_{r,n} = \ldots = a_r\,,$$

where a_r is the coefficient of x^r in $F(x)$. Hence

$$p_n(x) = \sum_{r=0}^{N} a_{r,n}\, x^r = \sum_{r=0}^{M} a_r\, x^r + \sum_{r=M+1}^{N} a_{r,n}\, x^r,$$

where $M < N$ and $M \to \infty$ as $N \to \infty$; and so

$$|F(x) - p_n(x)| = \left| \sum_{r=M+1}^{\infty} c_r\, x^r \right|,$$

where $a_r \geqslant c_r \geqslant 0$ since $0 \leqslant a_{r,n} \leqslant a_r$. Consequently

$$|F(x) - p_n(x)| \leqslant \left| \sum_{r=M+1}^{\infty} a_r\, x^r \right| \to 0$$

as $M \to \infty$, and so $p_n(x) \to F(x)$. Similarly

$$\lim_{n\to\infty} q_n(x) = G(x).$$

We now enunciate Legendre's theorem [**5**]:

LEMMA 3. *If $m_1, m_2, \ldots,$ and $n_1, n_2, \ldots$ are positive integers and $0 < m_i/n_i < 1$ for $i \geqslant i_0$, the continued fraction*

$$\frac{m_1}{n_1 +}\;\frac{m_2}{n_2 +}\;\frac{m_3}{n_3 +}\cdots$$

is irrational.

For a proof of this see, for example, [**3**]. This result has been improved upon, in particular by Bernstein and Szasz [**1**] who deduced the irrationality of the Jacobi theta series

$$\sum_{p=0}^{\infty} \left(\frac{r}{s}\right)^{p^2} \left(\frac{m}{n}\right)^{p}$$

when r, s, m, n are positive integers $s \geqslant 2$ and $0 < r^3 < s$. In the present paper we only make use of the result in its original form as no improvement is obtained by using any of the stronger forms.

3. Proof of Theorem 1. Let r, s be positive integers such that $(r, s) = 1$ and $r < s$. By Lemma 2,

$$H(x) = \frac{F(x)}{G(x)} = \frac{x^{\psi(1)}}{1+}\;\frac{x^{\psi(2)}}{1+}\;\frac{x^{\psi(3)}}{1+}\cdots.$$

Put $x = r/s$. Then from the well-known equivalence

$$\frac{b_1}{a_1+}\;\frac{b_2}{a_2+}\;\frac{b_3}{a_3+}\cdots = \frac{c_1 b_1}{c_1 a_1+}\;\frac{c_1 c_2 b_2}{c_2 a_2+}\;\frac{c_2 c_3 b_3}{c_3 a_3+}\cdots \qquad (c_i \neq 0)$$

we deduce, on taking

$$a_n = 1,\quad b_n = \left(\frac{r}{s}\right)^{\psi(n)},\quad c_1 = s^{\psi(1)},$$

and for $n \geqslant 1$,

$$c_n c_{n+1} = s^{\psi(n+1)},$$

that

$$H\left(\frac{r}{s}\right) = \frac{r^{\psi(1)}}{s^{\alpha_1}+}\;\frac{r^{\psi(2)}}{s^{\alpha_2}+}\;\cdots\;\frac{r^{\psi(n)}}{s^{\alpha_n}+}\;\cdots.$$

The successive exponents α_n satisfy

$$\alpha_1 = \psi(1),\quad \alpha_n = \psi(n) - \alpha_{n-1} \qquad (n \geqslant 2),$$

so that

$$\alpha_n = \sum_{r=0}^{n-1} (-1)^r \psi(n-r).$$

Now Lemma 3 is applicable if, for say $n \geqslant n_0$,

$$0 < r^{\psi(n)} < s^{\alpha_n},\quad \text{that is,}\quad 0 < r < s^{\gamma_n},$$

where

$$\gamma_n = \alpha_n/\psi(n) = \frac{1}{\psi(n)}\sum_{r=0}^{n-1} (-1)^r \psi(n-r)\,.$$

Let

$$\gamma = \liminf_{n\to\infty} \frac{1}{\psi(n)}\sum_{r=0}^{n-1} (-1)^r \psi(n-r)\,.$$

Evidently $\gamma \leqslant 1$. Then $r < s^{\gamma_n}$ for $n > n_0$ if $r < s^{\gamma}$ and so

$$(r/s^{\gamma_n})^{\psi(n)} < 1.$$

This proves Theorem 1.

If $\psi(n) - \psi(n-1)$ is an increasing function we can prove that $\gamma \geqslant \frac{1}{2}$. For if $\phi(n) = \psi(n) - \psi(n-1)$ $(n \geqslant 2)$ and $\phi(1) = \psi(1)$, then

$$\psi(n) = \sum_{r=1}^{n} \phi(r)\,.$$

Now

$$\gamma = \liminf_{n\to\infty} \frac{1}{\psi(n)}\sum_{r=0}^{n-1} (-1)^r \psi(n-r)\,,$$

and so

$$\gamma = \liminf_{n\to\infty} \frac{\phi(n) + \phi(n-2) + \phi(n-4) + \dots}{\phi(n) + \phi(n-1) + \phi(n-2) + \dots + \phi(1)} = \liminf_{n\to\infty} \gamma_n,$$

say, the last term in the numerator being $\phi(2)$ or $\phi(1)$ according as n is even or odd. Since $\phi(n)$ is an increasing function of n,

$$\phi(n)+\phi(n-2)+\phi(n-4)+\ldots>\phi(n-1)+\phi(n-3)+\phi(n-5)+\ldots$$

and so $\gamma_n > \frac{1}{2}$ for

$$\sum_{r=1}^{n}\phi(r)=\sum_{k\geqslant 0}^{2k\leqslant n-1}\phi(n-2k)+\sum_{k\geqslant 0}^{2k\leqslant n-2}\phi(n-2k-1),$$

that is,

$$\sum_{r=1}^{n}\phi(r)<2\sum_{k\geqslant 0}^{2k\leqslant n-1}\phi(n-2k).$$

Hence

$$\gamma=\liminf_{n\to\infty}\gamma_n\geqslant\tfrac{1}{2}.$$

4. Applications. The most interesting application is obtained by taking $\psi(n)=n$. In this case we have

$$\frac{1}{1+H(x)}=\frac{1}{1+}\;\frac{x}{1+}\;\frac{x^2}{1+}\;\frac{x^3}{1+}\;\cdots \tag{7}$$

which is the Rogers-Ramanujan continued fraction [**6**; **7**]. Now

$$\gamma=\liminf_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}(-1)^r(n-r)=\tfrac{1}{2}. \tag{8}$$

The Rogers-Ramanujan identities give

$$1+H(x)=\prod_{n=0}^{\infty}\frac{(1-x^{5n+2})(1-x^{5n+3})}{(1-x^{5n+1})(1-x^{5n+4})} \tag{9}$$

and from Theorem 1 we have at once:

THEOREM 2. *If r, s are positive integers satisfying $(r,s)=1, r^2<s$, the product*

$$\prod_{n=0}^{\infty}\frac{(s^{5n+2}-r^{5n+2})(s^{5n+3}-r^{5n+3})}{(s^{5n+1}-r^{5n+1})(s^{5n+4}-r^{5n+4})}$$

is irrational.

We now give another form of this result. Write

$$F(x)=\sum_{n=0}^{\infty}a_n x^n,\quad G(x)=\sum_{n=0}^{\infty}b_n x^n$$

where a_n is the number of partitions of n with least element 1 and minimal difference 2; $b_0=1$, and for $n\geqslant 1$, b_n is the number of partitions of n with least element not less than 2 and minimal difference 2. Then

$$F(x)+G(x)=\sum_{n=0}^{\infty}c_n x^n,$$

say, where $c_0=1$ and, for $n\geqslant 1$, c_n is the number of partitions of n with minimal difference 2. Now it is a consequence of the Rogers-Ramanujan identities [**4**]

that the number of such partitions is equal to the number of partitions of n into parts of the forms $5m + 1$ and $5m + 4$ and so

$$(10) \qquad F(x) + G(x) = \sum_{n=0}^{\infty} c_n x^n = \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-1}(1 - x^{5n+4})^{-1}.$$

Now, from (9)

$$(11) \qquad \prod_{n=0}^{\infty} \frac{(1 - x^{5n+2})(1 - x^{5n+3})}{(1 - x^{5n+1})(1 - x^{5n+4})} = 1 + H(x) = \frac{F(x) + G(x)}{G(x)}$$

so that (10) and (11) enable us to determine $F(x)$ and $G(x)$ in terms of infinite products, namely,

$$G(x) = \prod_{n=0}^{\infty} (1 - x^{5n+2})^{-1}(1 - x^{5n+3})^{-1},$$

$$F(x) = \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-1}(1 - x^{5n+4})^{-1} - \prod_{n=0}^{\infty} (1 - x^{5n+2})^{-1}(1 - x^{5n+3})^{-1}.$$

Next we note that (9) may be written in the form

$$(12) \quad 1 + H(x) = \prod_{n=1}^{\infty} (1 - x^n)(1 - x^{5n})^{-1} \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-2}(1 - x^{5n+4})^{-2},$$

and since the left side of (12) is irrational when $x = r/s$, $(r, s) = 1$, $r^2 < s$ at least one of the factors

$$\prod_{n=1}^{\infty} (1 - x^n), \quad \prod_{n=1}^{\infty} (1 - x^{5n}), \quad \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-2}(1 - x^{5n+4})^{-2}$$

is irrational. The product

$$\prod_{n=1}^{\infty} (1 - x^n)$$

is of importance in the theory of elliptic functions; it was considered by Euler **[2]** who proved that

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}.$$

We see then that if we write $p'(n)$ to denote the number of partitions of n into parts of the forms $5m + 1$ and $5m + 4$ we have the following alternative to Theorem 2:

THEOREM 3. *If $x = r/s$ where r, s are positive integers satisfying $(r, s) = 1$, $r^2 < s$ then at least one of the numbers*

$$\sum_{n=1}^{\infty} p'(n) x^n, \quad \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}, \quad \sum_{n=-\infty}^{\infty} (-1)^n x^{5n(3n+1)/2}$$

is irrational.

Two other cases of interest are given by

(i) $\psi(n) = 2^{n-1}$,

(ii) $\{\psi(n)\} = \{1, 1, 2, 4, 6, 10, \ldots\}$, where $\psi(n + 1) = \psi(n) + \psi(n - 1)$ for $n > 4$.

In case (i) $F(x) = xG(x^2)$ and since

$$\gamma = \liminf_{n\to\infty} \sum_{r=0}^{n-1} (-1)^r\, 2^{-r} = \tfrac{2}{3}$$

we obtain the following:

THEOREM 4. *If $G(x)$ is the function defined in Theorem 1 when $\psi(n) = 2^{n-1}$ and r, s are positive integers satisfying $(r, s) = 1$, $r^3 < s^2$ then at least one of the numbers $G(r/s)$, $G(r^2/s^2)$ is irrational.*

In case (ii), $F(x) = x/(1 - x^2)$ and so is rational for rational x. It follows from Theorem 1 that $G(x)$ and also $F(x) + G(x)$ is irrational when $x = r/s$, $(r, s) = 1$, $r < s^{\gamma}$. As is easily seen,

$$\psi(n) = A(\alpha^{n-1} - \beta^{n-1}) \qquad (n > 3)$$

where $2\alpha = 1 + \sqrt{5}$, $2\beta = 1 - \sqrt{5}$, $2A^{-1} = \sqrt{5}$. Hence

$$\begin{aligned} \gamma_n &= \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r) \\ &= \frac{(\alpha^{n-1} - \alpha^{n-2} + \alpha^{n-3} - \dots) - (\beta^{n-1} - \beta^{n-2} + \beta^{n-3} - \dots)}{\alpha^{n-1} - \beta^{n-1}}. \end{aligned}$$

Divide throughout by α^{n-1}, then since $|\beta| < \alpha$, we have

$$\gamma_n \to 1 - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^3} + \dots = \frac{\alpha}{1+\alpha} \qquad \text{as } n \to \infty$$

and so $\gamma = \alpha/(1 + \alpha) = -\beta$. Now

$$F(x) + G(x) = \sum_{p=0}^{\infty} (a_p + b_p)\, x^p = \sum_{p=0}^{\infty} c_p\, x^p$$

where $c_0 = 1$, and for $p > 1$, c_p is equal to the number of ways of expressing p as the sum of elements of the sequence 1, 1, 2, 4, 6, 10, 16, ... without repetitions and with no two consecutive elements occurring in the same decomposition. From Theorem 1 we have at once:

THEOREM 5. *If*

$$K(x) = \sum_{p=0}^{\infty} c_p\, x^p$$

is the power series just defined and r, s are positive integers satisfying $(r, s) = 1$, $r^2 < s^{\sqrt{5}-1}$ then the number $K(r/s)$ is irrational.

The details of the proofs of Theorems 4 and 5 are straightforward and I omit them.

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References

1. F. Bernstein and O. Szász, *Über Irrationalität unendlicher Kettenbrüche*, Math. Ann., *76* (1915), 295–300.
2. L. Euler, *Introductio in analysin infinitorum* (Lausanne, 1748), 270.
3. H. S. Hall and S. R. Knight, *Higher algebra* (London, 1882), 366.
4. G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers* (Oxford, 1938), 289.
5. A. M. Legendre, *Géométrie et trigonométrie* (Paris, 1821), 290.
6. S. Ramanujan, *Collected papers* (Cambridge, 1927), 214–215.
7. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. (1), *25* (1894), 329.

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STABLE LATTICES

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1. Introduction. The consideration of *relative extrema* to correspond to the *absolute extremum* which is the critical lattice has been going on for some time. As far back as 1873, Korkine and Zolotareff [**6**] worked with the ellipsoid in hyperspace (i.e., with quadratic forms), and later Minkowski [**8**] worked with a general convex body in two or three dimensions. They showed how to find critical lattices by selection from among a finite number of relative extrema. They were aided by the long-recognized premise that only a finite number of lattice points can enter into consideration [**1**] when one deals with lattices "admissible to convex bodies."

In the realm of the more general star body, such as that involved in the product of homogeneous forms, there is no finiteness principle of equal scope. Mahler [**7**] has, however, developed a local property for the critical lattices established by Davenport [**3**; **4**], by using "bounded reducibility."

We shall develop a similar property which we shall call *stability* and we shall extend it to a large class of lattices for star bodies. We shall do this by introducing a formalism of *positively dependent differentials*. Then as an illustration, we shall give a condition for stability of the norm in an algebraic module (a condition which can be considerably simplified by the use of units in the Dedekind order [**5**]).

2. Critical lattices. A lattice $\mathfrak{L}$, in n-space, is determined by a $n \times n$ matrix (a_{ij}) of (say) positive determinant $||a_{ij}||$, the points of the lattice being denoted by the vectors $\mathbf{x}$ with components

$$x_i = \sum_{j=1}^{n} a_{ij} m_j \qquad (i = 1, 2, \ldots, n) \tag{2.1}$$

corresponding to the integral n-tuple $(m_j) = (m_1, \ldots, m_n)$. A real continuously differentiable function, $\phi(\mathbf{x})$, homogeneous of positive degree h in its variables, is defined in the space. The locus $|\phi| = 1$ is the boundary of the star body under consideration. Then we consider the function defined by

$$F(m_j, \mathfrak{L}) = \phi(\mathbf{x})/||a_{ij}||^{h/n}. \tag{2.2}$$

For each lattice $\mathfrak{L}$, an infimum of $|F|$ is defined over (m_j), excluding the origin of course. Call it $M(\mathfrak{L})$. Let the values $M(\mathfrak{L})$ have a finite supremum M_0. Now if a lattice $\mathfrak{L}_0$ exists for which $M(\mathfrak{L}_0) = M_0$, then $\mathfrak{L}_0$ is called a *critical* lattice. Thus

$$M(\mathfrak{L}) = \inf |F(m_j, \mathfrak{L})| \qquad (m_j) \neq 0 \tag{2.3}$$

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and

$$M_0 = M(\mathfrak{L}_0) = \sup_{\mathfrak{L}} M(\mathfrak{L}). \tag{2.4}$$

The *stable* lattices $\mathfrak{L}^*$ will be presently defined using the geometrically simplest (rather than the most general) type of relative maximum to replace the (absolute) maximum required in the last formula.

3. Dimension of differentials. First of all let us consider the function F defined in formula (2.2). If it refers to the n-tuple $(m_j^{(k)})$ of a finite or infinite set indexed by k then it will be called $F^{(k)}$ $(= F(m_j^{(k)}, \mathfrak{L}))$ for short. Now we regard the a_{ij} as the variables, leading to differentials da_{ij} around a fixed lattice (a_{ij}); the (m_j) or $(m_j^{(k)})$ remaining fixed in this process. Thus a total differential $d|F|$ or $d|F^{(k)}|$ is defined. (Since we are excluding lattices where $M(\mathfrak{L}) = 0$, the presence of the absolute value sign creates no difficulties.)

We now define dimension: The set (finite or infinite) of differentials $d|F^{(k)}|$ is of *dimension* q if a certain subset of q differentials are linearly independent and provide a basis for all differentials. In particular, the set of $q+1$ differentials $d|F^{(k)}|$ $(k = 1, 2, \ldots, q+1)$ is of dimension q if and only if the values of $A^{(k)}$ for which

$$\sum_{1}^{q+1} A^{(k)}\, d|F^{(k)}| = 0 \tag{3.1}$$

are a set of dimension one (i.e., proportional to the components of a single non-zero vector $\mathbf{A}_0$).

We next consider the actual calculation of dimension. We write the sum

$$\sum_{(k)} A^{(k)}\, d|F^{(k)}| = 0 \tag{3.11}$$

over some finite set of indices. Clearly the $A^{(k)}$ must be independent of the coordinates describing the $F^{(k)}$, but in terms of the a_{ij}, for instance,

$$\sum_{(k)} A^{(k)} \frac{\partial|F^{(k)}|}{\partial a_{ij}} = 0, \qquad 1 \leq i, j \leq n, \tag{3.2}$$

where the partial derivatives, as agreed earlier, are evaluated at some fixed lattice (a_{ij}). But $F^{(k)} = \phi(\mathbf{x}^{(k)})/\Delta^{h/n}$ where the $x_i^{(k)}$ were given in formula (2.1) (with $m_i = m_i^{(k)}$) and $\Delta = \|a_{ij}\|$, assumed positive for convenience. Now

$$\frac{\partial|F^{(k)}|}{\partial a_{ij}} = \Delta^{-h/n} \frac{\partial|\phi^{(k)}|}{\partial x_j^{(k)}} m_i^{(k)} - \frac{h}{n}\Delta^{-(h/n)-1}|\phi^{(k)}|\, a^{ij}, \tag{3.3}$$

where a^{ij} means the cofactor of a_{ij} in the matrix (a_{ij}) and $\phi^{(k)}$ means $\phi(\mathbf{x}^{(k)})$. Thus we can take system (3.2), multiply by a_{il}, and sum over i, obtaining on substitution of the result in (3.3),

$$\begin{aligned} &\sum_{(k)} A^{(k)} R_{jl}^{(k)} = 0, \\ &R_{jl}^{(k)} = \frac{\partial|\phi^{(k)}|}{\partial x_j^{(k)}} x_l^{(k)} - \frac{h}{n}|\phi^{(k)}|\delta_{jl}, \end{aligned} \qquad 1 \leq j, l \leq n, \tag{3.4}$$

where δ_{jl} is the Kronecker delta. Since $\Delta \neq 0$, system (3.4) must be equivalent to system (3.2). Note, however, that system (3.4) does not contain a_{ij} or $m_i^{(k)}$ explicitly, and indeed the dimension can be determined as the rank of the $n^2 \times \Omega$ matrix of $R_{jl}^{(k)}$ where j, l takes on n^2 indices and k takes on Ω indices, finite or infinite.

Since, conversely, the system (3.4) leads to the system (3.11), the differentials depend essentially only on the lattice points $\mathbf{x}^{(k)}$.

4. Free dimension. Now if our set of $(m_j^{(k)})$ took on all *integral* n-tuples (except the origin), a dimension would be defined for these $d|F^{(k)}|$. Its value will be called *the free dimension of the fixed lattice* (a_{ij}) *with respect to the function* $\phi(\mathbf{x})$. We shall now see that the free dimension depends only on ϕ and not on the a_{ij}.

First of all the definition of free dimension would not change if the $(m_j^{(k)})$ took on all *real* values (except the origin). To see this, we must ask if the rank of the system $R_{jl}^{(k)}$ becomes any *greater* if the $(m_j^{(k)})$ are real instead of integral. Suppose a certain $q \times q$ minor of $R_{jl}^{(k)}$ is non-vanishing for a real set of $(m_j^{(k)})$. By homogeneity this depends on the $n - 1$ ratios of the n components of each $(x_i^{(k)})$ $(i = 1, 2, \ldots, n)$; but certainly any such ratios can be approximated arbitrarily closely by ratios from n components of integral lattice points. This is a simple consequence of the Dirichlet boxing-in principle. Hence this same $q \times q$ minor will be non-vanishing for an integral set of $(m_j^{(k)})$.

Thus in determining the *free dimension* from system (3.4) we may regard $R_{jl}^{(k)}$ as a function of the *free variables* $\mathbf{x}^{(k)}$ no longer subjected to membership in a lattice. The coefficients $A^{(k)}$, for instance, can be taken as polynomials in $x_i^{(k)}$ when ϕ is a rational function, by virtue of the fact that a polynomial in several variables vanishes for all values of the variables only when it vanishes identically. The free dimension will be denoted by Q.

Our only general information about ϕ is that it is homogeneous; hence in formula (2.2) one of the variables a_{ij} can be cancelled out, making F dependent on only $n^2 - 1$ of them. In a corresponding way, from formula (3.4),

$$\sum_{jl} R_{jl}^{(k)} \delta_{jl} = 0$$

by Euler's theorem on homogeneous functions. Hence, clearly, $Q < n^2 - 1$.

From the theory of implicit functions it follows that the free dimension is the minimum number of variables, on which the various $F^{(k)}$ really depend as the a_{ij} vary, the $(m_j^{(k)})$ remaining fixed. (See equation (3.2).)

5. Stable lattices. We first define a set of vectors $\{\mathbf{g}^{(k)}\}$ $(k = 1, 2, \ldots, Q + 1)$ to be *positively dependent* if all the linear relations

$$\sum_{(k)} A^{(k)} \mathbf{g}^{(k)} = 0 \tag{5.1}$$

are such that, $A^{(k)} = \sigma A_0^{(k)}$, where $A_0^{(k)}$ is a set of $Q + 1$ positive numbers and σ is scalar. This definition is affine-invariant and it can be realized only if

the set of vectors has dimension Q. An equivalent definition is that an arbitrary vector in $\mathfrak{S}^Q$, the Q-dimensional space determined by the $\{\mathfrak{g}^{(k)}\}$, will have a positive projection (in the sense of any ordinary scalar product) on at least one $\mathfrak{g}^{(k)}$.

We now come to the major definition. Assume the function ϕ to have the free dimension Q. A lattice $\mathfrak{L}^* = (a^*_{ij})$ is called *stable with respect to* ϕ if $Q+1$ n-tuples $(m_i^{(k)})$ exist such that

$$|F(m_j^{(k)}, \mathfrak{L}^*)| = M(\mathfrak{L}^*) > 0 \qquad (k = 1, 2, \ldots, Q+1) \tag{5.2}$$

and

$$\text{the } d|F^{(k)}| \text{ are positively dependent at } \mathfrak{L}^*. \tag{5.3}$$

We next develop the most important property of the stable lattice. For the free dimension Q, we can, by the implicit function theorem select $n^2 = Q + t$ variables

$$b_1, \ldots, b_Q, c_1, \ldots, c_t$$

functionally equivalent to the a_{ij} and such that the $|F^{(k)}|$ actually depend (locally) on the b_i only, (i.e., $\partial|F^{(k)}|/\partial c_j \equiv 0$), for all non-zero n-tuples $(m_i^{(k)})$, integral or even real (see §4). Having found such variables, we describe the $F^{(k)}$ locally in terms of just the b_i, in the neighborhood of b^*_i, c^*_j (the variables corresponding to a^*_{ij}). We define

$$|\mathfrak{L} - \mathfrak{L}^*| = \left(\sum_{i=1}^{Q} (b_i - b^*_i)^2\right)^{\frac{1}{2}} \tag{5.4}$$

as the *distance between lattices* (corresponding to the local coordinate system). Then if a lattice $\mathfrak{L}^* = (a^*_{ij})$ is stable with respect to the function ϕ of free dimension Q, then an $\epsilon > 0$ and an $\eta > 0$ exist together with $Q+1$ n-tuples $(m_i^{(k)})$ $(k = 1, 2, \ldots, Q+1)$ such that

$$|F(m_i^{(k)}, \mathfrak{L}^*)| = M(\mathfrak{L}^*) \tag{5.2}$$

and

$$|F(m_i^{(k)}, \mathfrak{L})| < |F(m_i^{(k)}, \mathfrak{L}^*)| - \eta|\mathfrak{L} - \mathfrak{L}^*| \tag{5.31}$$

for some index k depending on $\mathfrak{L}$ providing only that $|\mathfrak{L} - \mathfrak{L}^*| < \epsilon$.

Conversely, if a lattice has these properties for some ϵ, η and $Q+1$ n-tuples as described, then the lattice is stable.

This is the type of property that Mahler [7] established for certain cases where his theory of "bounded reducibility" applies. The property is an easy consequence of the definition of stability given above. As a simple corollary we note that when $\mathfrak{L}^*$ is stable, then for some positive ϵ, η, and all $|\mathfrak{L} - \mathfrak{L}^*| < \epsilon$, we have $M(\mathfrak{L}) < M(\mathfrak{L}^*) - \eta|\mathfrak{L} - \mathfrak{L}^*|$.

Thus, in a sense, $M(\mathfrak{L}^*)$ is a local maximum by virtue of the $Q+1$ specific lattice points for which the value is assumed.

6. Positive span. Now there are often infinitely many vectors of a lattice satisfying condition (5.2) for stability. Hence it would seem precarious to expect

to discover $Q + 1$ vectors satisfying the further condition (5.3). We shall simplify the criterion of stability by referring it to the aggregate of *all* vectors satisfying the condition (5.2).

Let the finite or infinite set of vectors $\{\mathbf{w}\}$ determine a space $\mathfrak{S}^Q$ of Q dimensions, in which unit basis vectors and any ordinary scalar product are introduced. Then the set $\{\mathbf{w}\}$ is said to *positively span* this space if it has both the (*projection*) property that every arbitrary vector of $\mathfrak{S}^Q$ has a positive projection on at least one vector of $\{\mathbf{w}\}$ and the (*independence*) property that every Q vectors of $\{\mathbf{w}\}$ are linearly independent. (The properties are easily seen to be affine-invariant.)

Now every set of $Q + 1$ positively dependent vectors will positively span its space $\mathfrak{S}^Q$. (See §5.) Conversely, *every set of vectors* $\{\mathbf{w}\}$ *that positively spans the space* $\mathfrak{S}^Q$ *contains* $Q + 1$ *vectors which are positively dependent.* To see this, note that the set $\{\mathbf{w}\}$ determines a set of end-points of the various vectors, whose convex closure, a polytope $\mathfrak{P}$ in $\mathfrak{S}^Q$, contains the origin by the projection property. (The set $\{\mathbf{w}\}$ may be considered as a finite set by the compactness of the hypersphere in $\mathfrak{S}^Q$.) But the polytope $\mathfrak{P}$ can then be reduced to a single simplex of $Q + 1$ vertices which contains the origin, if we merely triangulate $\mathfrak{P}$, introducing no new vertices and recalling that the triangulating hyperplanes will not contain the origin (by the independence property). These $Q + 1$ vertices of course determine the positively dependent vectors.

The applications that follow (see §9) will stem from the following result: Let the (infinite) set of vectors $\{\mathbf{v}\}$ of $\mathfrak{S}^Q$ contain vectors arbitrarily close in direction to $2Q$ vectors consisting of Q basis vectors and their negatives. We shall call this latter configuration a *unit star.* (Compare the "eutactic star" of Coxeter [**2**, p. 401].) Suppose that a subsequence of $\{\mathbf{v}\}$ can be selected which comes increasingly close in direction to any designated one of the $2Q$ vectors of the unit star, without coinciding in direction. Let us further suppose that the last property still holds if we exclude from consideration all vectors of $\{\mathbf{v}\}$ that lie in an arbitrary set of fixed hyperplanes (through the origin). Then *the set* $\{\mathbf{v}\}$ *has a subset that positively spans* $\mathfrak{S}^Q$ *and hence a further subset of* $Q + 1$ *positively dependent vectors.*

To see this, first note that the *projection* property is easy to obtain from the unit star. To establish the *independence* property we select as our subset of $\{\mathbf{v}\}$ the sequence of vectors chosen by the following inductive procedure: The vectors are to come increasingly close to the $2Q$ vectors of the unit star in some order (without coinciding). Furthermore each vector is selected to be linearly independent of the set consisting of earlier chosen vectors (if any) together with the $2Q$ vectors of the unit star. This process will at every step leave cones about the vectors of the unit star inside one of which the next vector of the sequence may be chosen.

7. Stability of the norm in a module. Probably the most investigated non-convex star body is that given by $\phi(\mathbf{x}) = x_1 x_2 \dots x_n$ and the most important lattices for it are those for which each row of the matrix in (2.1) namely

$a_{i1}, a_{i2} \ldots, a_{in}$, is the i conjugate of a set of n basis numbers of a (non-singular) module $\mathfrak{M}$ in a totally real field of degree n. The x_i is the i conjugate of a general number $\mathbf{x}$ in the module and ϕ is the norm. The minimum absolute value of ϕ (when $(x_i) \neq (0)$), is of course actually achieved for some $\mathbf{x}$ and is denoted by $M'(\mathfrak{M}) > 0$.

Now for this ϕ, from (3.4), $R_{j,l}^{(k)} = |\phi^{(k)}| \, x_l^{(k)}/x_j^{(k)}$ when $j \neq l$, and 0 when $j = l$. Thus Q, the free dimension, is $n(n-1)$ according to either of two reasons given in §4. *The condition for stability of the norm is simply that some set of vectors* $\mathbf{x}^{(k)}$ *of the module* $\mathfrak{M}$ *is such that*

$$\text{norm } \mathbf{x}^{(k)} = \pm M'(\mathfrak{M}) \tag{7.1}$$

and the set of vectors indexed by k and with $n(n-1)$ components

$$(\ldots, x_l^{(k)}/x_j^{(k)}, \ldots) \qquad (l, j = 1, 2, \ldots, n; l \neq j), \tag{7.2}$$

positively spans a space of $n(n-1)$ dimensions.

If the field has r real conjugates and $2s$ complex conjugates $(r + 2s = n)$, then we proceed similarly, writing

$$\begin{aligned} z_j &= x_j && (j = 1, 2, \ldots, r), \\ z_j &= x_j + ix_{j+s} && (j = r+1, \ldots, r+s), \\ z_{j+s} &= x_j - ix_{j+s}, && \end{aligned} \tag{7.3}$$

where z_i represents the n conjugates of a number $\mathbf{z}$ in the field. Then the norm is

$$\phi^*(\mathbf{x}) = x_1 \ldots x_r(x_{r+1}^2 + x_{r+s}^2) \ldots (x_{r+s+1}^2 + x_{r+2s}^2)$$

and the lattice (or module $\mathfrak{M}$) is this time determined by the n^2 different real and imaginary components of the matrix a_{ij}. Then a repetition of the earlier calculation yields the same Q and the convenient new *condition for stability of the norm as simply that some set of vectors $\mathbf{z}^{(k)}$ of the module $\mathfrak{M}$ is such that*

$$\text{norm } \mathbf{z}^{(k)} = \pm M'(\mathfrak{M}), \tag{7.11}$$

and the set of vectors, indexed by k and given by the following $2n(n-1)$ components:

$$(\ldots, \Re\, z_l^{(k)}/z_j^{(k)}, \Im\, z_l^{(k)}/z_j^{(k)}, \ldots) \qquad (l, j = 1, 2, \ldots, n; l \neq j), \tag{7.21}$$

positively spans a space of $n(n-1)$ dimensions.

Of these $2n(n-1)$ components, some may vanish and others may be repeated with or without change of sign. Taking this into account we find the total, in general, is still $Q = n(n-1)$ essentially distinct non-vanishing components.

8. Quadratic forms. A simple illustration of stable lattices can be given for the indefinite form $\phi = x_1 x_2$ of free dimension $Q = 2$. (Compare §7.) Expanding $\phi = \Phi(m_1, m_2)$, in the notation of (2.1) and (2.2) we obtain:

$$\begin{aligned} F &= \Phi(m_1, m_2)/d^{\frac{1}{2}}, \\ \Phi(m_1, m_2) &= am_1^2 + bm_1 m_2 + cm_2^2, \qquad d = b^2 - 4ac > 0. \end{aligned} \tag{8.1}$$

The lattice basis is involved by means of the relations

$$(8.2)\qquad \begin{aligned} a_{11}a_{21} &= a, & a_{12}a_{22} &= c,\\ a_{11}a_{22}+a_{12}a_{11} &= b, & d &= \Delta^2. \end{aligned}$$

Following §7, we would restrict ourselves to the case where Φ is proportional to the norm function in a quadratic module, or a, b, c are rational and d is not a perfect square. (Actually this would follow automatically from the stability conditions (5.2) and (5.3).) At any rate we assume Φ to be stable and let $Q+1=3$ integer couples $(m_1^{(k)}, m_2^{(k)})$ $(k=1,2,3)$ occur for which $|F|$ assumes its minimum value M, that is,

$$(8.3)\qquad F^{(k)} = \epsilon^{(k)}M \neq 0, \qquad (\epsilon^{(k)} = \pm 1).$$

Then we consider the vector space spanned by the vectors of two components

$$(8.4)\qquad R_{jl}^{(k)} = |\phi^{(k)}|x_l^{(k)}/x_j^{(k)} \qquad (j,l) = (1,2),(2,1).$$

In particular we look for relations of the type

$$(8.5)\qquad \sum_{k=1}^{3} A^{(k)}R_{jl}^{(k)} = 0,$$

valid for both of the above choices of (j, l). But by virtue of relation (8.3), or $x_1^{(k)}x_2^{(k)} = d^{\frac{1}{2}}M\epsilon^{(k)}$, it follows that relations of the type (8.5) are the same as the following conditions for stability:

$$(8.6)\qquad \sum_{k=1}^{3} A^{(k)}\mathbf{W}^{(k)} = 0,$$

where $\mathbf{W}^{(k)} = \epsilon^{(k)}((x_1^{(k)})^2, (x_2^{(k)})^2)$ is a vector of two components.

It is well known that the *critical* lattice for $\phi = x_1x_2$ belongs to the form

$$(8.7)\qquad \Phi_0(m_1, m_2) = m_1^2 + m_1m_2 - m_2^2 = (m_1 - \theta_0 m_2)(m_1 - \theta'_0 m_2)$$

where $\theta_0, \theta'_0 = \frac{1}{2}(-1 \pm \sqrt{5})$, $d = 5$. For the corresponding lattice, $M(\mathfrak{L}_0) = 1/\sqrt{5}$, in the notation of (2.3). To verify that this $\mathfrak{L}_0$ is *stable* we make the choice, by trial and error: $(m_1^{(k)}, m_2^{(k)}) = (1,0), (0,1), (1,1)$ for $k = 1,2,3$, respectively. Here

$$\epsilon^{(1)} = \epsilon^{(3)} = -\epsilon^{(2)} = 1; \quad x_1^{(k)} = m_1^{(k)} - \theta_0 m_2^{(k)}, \quad x_2^{(k)} = m_1^{(k)} - \theta'_0 m_2^{(k)}.$$

It is now easy to verify that from the equations in (8.6), $(A^{(1)}, A^{(2)}, A^{(3)}) = \sigma(1,3,1)$, for a scalar σ, whence the three $\mathbf{W}^{(k)}$ (or ultimately $d|F^{(k)}|$) are *positively dependent*. This establishes stability. From the results of §9, it will turn out that the presence of both positive and negative $\epsilon^{(k)}$ is actually necessary and sufficient. The result can be expressed as follows:

Let m be the smallest positive number which an indefinite irreducible quadratic form with integral coefficients represents in absolute value. Then the corresponding lattice is stable if and only if the form represents both $+m$ and $-m$.

In showing the lattice $\mathfrak{L}_0$ (of (8.7)) to be stable with respect to $\phi = x_1 x_2$, we showed according to §5 that a neighbourhood of $\mathfrak{L}_0$ in lattice space exists in which $M(\mathfrak{L}) < 1/\sqrt{5}$. If we ask about the size of this neighbourhood we find that, fortunately, it includes *all* lattices, provided we take into account images of the neighbourhood under change of basis. In other words, $\mathfrak{L}_0$ *is the critical lattice.* To see this, we set

$$F = x_1 x_2/(\theta - \theta'), \qquad 1 > \theta > 0 > \theta', \tag{8.8}$$

where $x_1 = m_1 - \theta m_2$, $x_2 = m_1 - \theta' m_2$. Here θ and θ' represent the two degrees of freedom of the lattice. We now go back to the proof of stability of $\mathfrak{L}_0$, and we use the same $(m_1^{(k)}, m_2^{(k)})$ and the corresponding $(x_1^{(k)}, x_2^{(k)})$ for $k = 1, 2, 3$. Then clearly,

$$\begin{cases} |F^{(1)}| < 1/\sqrt{5}, & \theta - \theta' > \sqrt{5}, \\ |F^{(2)}| < 1/\sqrt{5}, & \theta - \theta' > -\sqrt{5}\,\theta\theta', \\ |F^{(3)}| < 1/\sqrt{5}, & \theta - \theta' > \sqrt{5}\,(1-\theta)(1-\theta'), \end{cases} \tag{8.9}$$

and by drawing the $\theta\theta'$ plane we see that at least one of the three right-hand inequalities will hold at every point of the region $0 < \theta < 1$, $\theta' < -1$ (a rather ample neighbourhood of the critical values θ_0, θ'_0). But every lattice is equivalent under change of basis to one lying in this neighbourhood, by an extension of Gauss's criterion for reduction, i.e., that one root and the negative reciprocal of its conjugate lie between 0 and 1. Thus the lattice $\mathfrak{L}_0$ is critical.

If we turn our attention to the definite form $\phi = x_1^2 + x_2^2$ we find that ϕ is stable only for the critical (equilateral) lattice, in which case ϕ is proportional under rotation, to the norm function of the integers in the field of the cube roots of unity. For the larger problem of $\phi = x_1^2 + \ldots + x_n^2$, analogous types of extrema have been developed [**6**; **9**; **2**] manifesting themselves largely in the *coefficients of the expanded form.* (The reader is referred to [**2**] for recent developments as well as a sizable bibliography.) The present paper, however, will conclude with a treatment of the norm function of §7 as a *factored form*; thus we shall finally arrive at a criterion of stability in terms of the *multiplicative arithmetic* of a field.

9. Application of units of the order. The module $\mathfrak{M}$ determines [**5**] another (non-singular) module $\mathfrak{O}$, called its *order*, which consists of all algebraic integers $\mathbf{v}$ (in the corresponding field) such that $\mathbf{v}\mathfrak{M}$ lies in $\mathfrak{M}$. For instance, if $\mathfrak{M}$ were the module of all integers in a field (called an *integer-module* for short), then we should have $\mathfrak{O} = \mathfrak{M}$ and the minimizing $\mathbf{x}$ (or $\mathbf{z}$) satisfying equations (7.1) (or (7.2)) would be the units. In any case, $\mathfrak{O}$ is also a ring with unity, and according to the classical theory it contains $r + s - 1$ fundamental units. We shall denote any unit by $\mathbf{u}$ (with conjugates u_i). Thus for every solution $\mathbf{x}$ (or $\mathbf{z}$) to (7.1) (or (7.11)) another one is of the type $\mathbf{xu}$ (or $\mathbf{zu}$). We shall now use these units to make the vectors $\mathbf{R}$ follow a *unit star.* (See §6.)

For instance we start by assuming $\mathfrak{M}$ to be totally real and we concentrate on the (j, l) component of $\mathbf{R}$, whose sign agrees with x_l/x_j. (See §7.) By using fundamental units, we can choose a variable unit $\mathbf{u}$ so that of the $n(n-1)$ components u_q/u_p $(p \neq q)$, the fixed component u_l/u_j is positive and of *greater order of magnitude* than any other. This guarantees the *projection property*, i.e., that for the new minimal vector $\mathbf{xu}$, the new vector $\mathbf{R}$ comes arbitrarily close in direction to

$$(0, \ldots, 0, \operatorname{sgn} x_l/x_j, 0, \ldots, 0),$$

the "sgn" being at the (j, l) component. We can go further. Along with the last condition on the variable $\mathbf{u}$, we can have each u_q/u_p approaching 0 or ∞ with a *different order of magnitude*. This will guarantee the *independence property*, i.e., the vectors $\mathbf{R}$ can not then all lie on a finite set of hyperplanes through the origin (as the components are now of different orders of magnitude). Thus the positive span is established provided the "sgn" can be made positive and negative.

The norm in a totally real module is therefore stable if and only if the numbers $\mathbf{x}$ *of minimal absolute norm* $(\neq 0)$ *are such that, for any two different conjugate fields denoted by* l *and* j, *the ratio* x_l/x_j *is positive for some* $\mathbf{x}$ *and negative for others.*

As an immediate consequence of this criterion, the integer-module of a *real quadratic field* is stable if and only if there is a unit of norm -1. The integer-module of a *totally real cubic field* is stable if and only if the units display all possible 2^3 arrays of sign among their conjugates. For instance the totally real cubic field generated by $2 \cos 2\pi/7$ (of minimal discriminant $= 49$) easily has the property, as we see by using as units conjugates of this generating element. Hence the field in question is stable [**7**].

In the case of modules which are not totally real it is harder to find a criterion as elegant as the last one, because $|u_l/u_j| = |u_q/u_p|$ not only when j and l refer to the *same* fields (respectively) as do p and q, but also when they refer to *conjugate complex* fields. Thus it is not so easy to accentuate just one component of $\mathbf{R}$. But since we are interested in the real or imaginary components of u_l/u_j (see (7.21)), we can make use of the incommensurability of π with the arguments of certain complex units. Thus by a modification of the unit star method as just used for the totally real case, we can see that *all cubic fields that are not totally real* have integer-modules with stable norms. Mahler [**7**] gave an analogous result only for the cubic field of (minimal) discriminant 23, but it curiously enough requires no special properties of that field!

Concluding in a simpler vein, we can see that if an integer-module is not totally real, but nevertheless has as units only totally real numbers (or even pure imaginary numbers) then it can not be stable, as the number of distinct non-zero components in (7.21) will fall short of the desired total, $Q = n(n-1)$, by at least s. Thus, once more, the *complex quadratic fields* do not have stable integer-modules, except when the field is that of the cube roots of unity.

Many other fields have been tested for stability but we shall defer the details for a later occasion as the techniques involved are rather specialized for the present stage of development.

References

1. H. Cohn, *On the finite determination of critical lattices*, Proc. Amer. Math. Soc., *2* (1951), 547–549.
2. H. S. M. Coxeter, *Extreme forms*, Can. J. Math., *3* (1951), 391-441.
3. H. Davenport, *On the product of three homogeneous linear forms* II, Proc. London Math. Soc. (2), *44* (1938), 412–431.
4. ——— *On the product of three homogeneous linear forms* III, Proc. London Math. Soc., *45* (1939), 98–125.
5. P. G. Dirichlet, *Vorlesungen über Zahlentheorie* (Braunschweig, 1894), 505 ff.
6. A. Korkine and G. Zolotareff, *Sur les formes quadratiques positives*, Math. Ann., *11* (1877), 242–292.
7. K. Mahler, *Lattice points in n-dimensional star bodies* II, Proc. Koninklijke Nederlandsche Akademie van Wetenschappen, *49* (1946), 524–532, 622–631.
8. H. Minkowski, *Diophantische Approximationen* (Berlin, 1907), 51–58.
9. G. Voronoï, *Recherches sur les paralléloèdres primitifs*, J. Reine Angew. Math., *134* (1908), 198–287.

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CORRECTION TO "AN EXTENSION OF MEYER'S THEOREM ON INDEFINITE TERNARY QUADRATIC FORMS"

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Mr. G. L. Watson of University College, London showed, by a counter-example, the falsity of Theorem 6 in my paper entitled "An extension of Meyer's Theorem on Indefinite Ternary Quadratic Forms" which appeared in this Journal, vol. 4 (1952), pp. 120–128. The error in Theorem 6 stemmed from an error in Theorem 2 which should read as follows:

THEOREM 2. *If the form f above is properly primitive, if* $(\Omega, \Delta) = p$, $\Omega \not\equiv 0 \not\equiv \Delta \pmod 4$ *and if p^3 does not divide $|g|$, and if there is an integer q prime to p and satisfying the following conditions*:

(i) *q is an odd prime or double an odd prime,*

(ii) *$-\Omega q$ is represented by the reciprocal form of g,*

(iii) *every solution of the congruence*

$$x^2 - qy^2 \equiv 1 \pmod p \tag{3}$$

is congruent (mod p) *to a solution of the Pell equation*

$$x^2 - qy^2 = 1, \tag{4}$$

then the form f is in a genus of one class.

The condition that f be properly primitive is probably not essential. The essential correction is in condition (ii) for which $(\Omega, \Delta) = p$ is necessary.

First we deal with Theorem 2 as altered and then indicate the other necessary changes in the paper. The purpose of condition (ii) is to show that G, the matrix of g, can be written in the form

$$\begin{bmatrix} \Omega B & \Omega b_1 \\ \Omega b_1^T & b \end{bmatrix}, \text{ replacing } \begin{bmatrix} pB & pb_1 \\ pb_1^T & b \end{bmatrix}$$

of page 122.

To show that the new condition (ii) assures this, notice first that the invariants Ω and Δ/p of G are relatively prime by the conditions of the theorem, and f properly primitive implies [**2**, Theorem 41] that we may consider

$$g \equiv \Omega a x^2 + \Omega b y^2 + (\Delta/p)cz^2 \pmod{8\Omega^4\Delta^2p^k}.$$

Hence the reciprocal form g_0 of g is congruent to

$$bc(\Delta/p)x^2 + ac(\Delta/p)y^2 + \Omega ab z^2.$$

Thus if g represents a binary form $\Omega\phi$ where $|\phi| = -q$, its reciprocal form represents $-q\Omega$ and conversely.

Lemmas 1, 2, 3 are proved without any but trivial alterations, which establish Theorem 2 as stated above since Theorem 1 shows that g is in a genus of one class. To Corollary 1 should be added the condition:

$$f \text{ properly primitive and } (\Omega, \Delta) = p.$$

To make Theorem 4 apply to the new situation we must add the condition that the reciprocal form of g represents $-\Omega q$ in all $R(r)$ for prime divisors, r, of Ω. It is not hard to see [**2**, Theorem 34 and the remark] that this condition is, in terms of the form g given above,

$$c_r(g_0) = (-|g_0|, \Omega q)_r, \text{ whenever } \Omega q\, |g_0| \text{ is a square in } R(r)$$

[**2**, Corollary 14], c_r being the Hasse symbol. Since $|g_0| = \Omega(\Delta/p)^2$, the condition reduces to $c_r(g_0) = (q|r)$ whenever r occurs in Ω to an odd power and q is a square in $R(r)$, that is,

(iv) $c_r(g_0) = 1$ if $(q|r) = 1$ and r occurs in Ω to an odd power.

Then Theorem 4, to apply to our situation, should read as follows:

THEOREM 4. *Let p be a fixed odd prime and f a properly primitive quadratic form for which $(\Omega, \Delta) = p$, neither Ω nor Δ being divisible by 4 or p^2, and g its p-related form. Then the reciprocal form of g represents $-\Omega q$ if and only if it represents it in $R(r)$ for all prime divisors of $2\Delta/p$ and* (iv) *holds for each r occurring to an odd power in Ω.*

Condition (iv) then must be added to Theorem 5.

Since, in Theorem 6, $p = 3$, $(q|3) = 1$, we must add the condition:

$$c_r(g_0) = 1$$

or its equivalent above for all r occurring in Ω to an odd power.

In the two examples given in my paper, this condition holds and the result stated is correct. The example given by Mr. Watson,

$$f = 2x^2 + 12y^2 + 6yz + 12z^2,$$

is, in the first place, improperly primitive and hence barred by the above discussion, but it has been indicated that this restriction is probably not essential. However, the form g_0 for this form is

$$5x^2 + 8y^2 - 4yz + 8z^2$$

and here $c_3(g_0) = -1$. Hence the essential condition is not met.

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